Application of the Choquet Integral in Multicriteria Decision Making

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Abstract

In this paper, we introduce the Choquet integral as a general tool for dealing with multiple criteria decision making. After a theoretical exposition giving the fundamental basis of the methodology, practical problems are addressed, in particular the problem of determining the fuzzy measure. We give an example of application, with two different approaches, together with their comparison.

1 Introduction

Since its introduction in 1974 by Sugeno [30], the concept of fuzzy measure has been often used in multicriteria decision making. The very first application, by Sugeno himself, was about evaluation of faces of women [30], and has been followed by many others in Japan from the eighties till now (see a selection of these applications up to 1994 in [13] and in a shortened version in [8]).

Despite the somewhat empirical way fuzzy measures were used at the beginning, it was very early noticed that fuzzy measures can model in some sense a kind of interaction between criteria (see e.g. the attempt of Ishii and Sugeno [18]), but this issue was not formalized till the proposal by Murofushi and Soneda [24] of an interaction index for a pair of criteria. Later, Grabisch proposed a generalization of this index [9] to any subset of criteria, and Grabisch and Roubens proposed an axiomatic basis for the interaction index [17], giving a consistent basis for dealing with the notion of interaction. This work around interaction enlightens the link between multicriteria decision making and cooperative game theory, which was already remarked by Murofushi in 1992 [23].

Till the beginning of the nineties, the Sugeno integral was used as the aggregation tool for computing an average global score, taking into account the importances of criteria expressed by a fuzzy measure. Then, after the proposal of Murofushi and Sugeno [32, 25] to use the Choquet integral [1], which is an extension of classical Lebesgue integral, —and thus of the well-known weighted sum—, it was quickly adopted among practitioners. Later, the properties of Choquet and Sugeno integrals as an aggregation operator were studied in depth, and their connection with OWA operators in their usual additive form [36] or weighted minimum form [5, 2] discovered (see e.g. [6, 15, 26] and the paper of Marichal in this book).

A difficulty which has slowed down the application of fuzzy measures is its exponential complexity, since one has to define a real number for each subset of the set of criteria, and also to find a mean of evaluating these numbers, either by expert elicitation (but then comes the problem of the real meaning of these numbers, not to speak of the burden on the decision maker), or by optimization. This is the reason why most of the time in applications, particular cases of fuzzy measures were used, needing only the definition of a "distribution" (linear in complexity) and a parameter, such as decomposable measures [35], λ-measures [30, 31], possibility measures [37, 4], etc. However, in the field of multicriteria decision making, such simple measures are unable to express non homogeneous interaction phenomena between criteria, as it becomes evident with the interaction index formalism. This is the reason why Grabisch has proposed the concept of k-additive measure, bridging the gap between decomposable measures and ordinary fuzzy measures [9, 11]. Interestingly enough, it happens that there is a strong link between k-additive measures and the interaction index.

In this paper, we will explain the methodology of using the Choquet integral in multicriteria decision making. Our exposition will avoid the listing of all properties of the Choquet integral and its relation with ordinary aggregation operations, since many publications have been done along this line, to which the interested reader may refer (basically, see [13] Chap. 8, and [8, 15]). However, we will detail more fundamental and practical aspects, in particular, the connection with game theory, and how to identify in an experimental problem the fuzzy measure modelling the decision maker's behaviour. Two methods will be explained, and illustrated in detail on a practical (although fictitious) example.

In the whole paper, we will work on a finite universe X of n elements (criteria). $\mathcal{P}(X)$ is the power set of X, while $|A|$ denotes the cardinal of a set A, A^c its complement, and $A \setminus B$ denotes the set difference. $\wedge, \vee$ denote min and max respectively.

Finally, we just mention that this paper does not aim to cover all the range of a multicriteria decision problem, but merely to address the aggregation step. It is known that for most of the methodologies, an aggregation step exists, but the quantities to be aggregated may be of different kinds (mainly scores, degrees of satisfaction, degrees of preference, preference relations, etc.), which can be numerical or simply qualitative (ordinal). For the sake of simplicity, but without loss of generality, it is assumed that we deal with numerical scores on criteria, expressed on a $[0,1]$ scale, supposed to be an interval scale. The aggregation of ordinal scores is a completely different topic, which will be not addressed here.

2 Basic material on fuzzy measures and Choquet integral

In order to be as far as possible self-contained, we give in this section necessary definitions, adapted for multicriteria decision making, and thus slightly less general. See the companion paper in this book on k-additive measures for more details.

Definition 1 *(Sugeno [30]) A* fuzzy measure *μ on X is a function $\mu : \mathcal{P}(X) \longrightarrow [0,1]$, satisfying the following axioms.*

(i) $\mu(\emptyset) = 0$.

(ii) $A \subset B \subset X$ *implies* $\mu(A) \leq \mu(B)$.

We will assume here $\mu(X) = 1$ as usual, although this is not necessary in general.

Definition 2 *(Choquet [1]) Let μ be a fuzzy measure on X, whose elements are denoted $x_1, \ldots, x_n$ here. The* discrete Choquet integral *of a function $f : X \longrightarrow \mathbb{R}^+$ with respect to μ is defined by*

$$\mathcal{C}_\mu(f) := \sum_{i=1}^{n} (f(x_{(i)}) - f(x_{(i-1)}))\mu(A_{(i)}), \tag{1}$$

where $\cdot_{(i)}$ indicates that the indices have been permuted so that $0 \leq f(x_{(1)}) \leq \cdots \leq f(x_{(n)})$, and $A_{(i)} := \{x_{(i)}, \ldots, x_{(n)}\}$, and $f(x_{(0)}) = 0$.

Definition 3 *Let μ be a set function (not necessarily a fuzzy measure) on X. The* Möbius transform *of μ is a set function on X defined by*

$$m(A) := \sum_{B \subset A} (-1)^{|A \setminus B|} \mu(B), \quad \forall A \subset X. \tag{2}$$

The transformation is inversible, and μ can be recovered from m by

$$\mu(A) = \sum_{B \subset A} m(B), \quad \forall A \subset X. \tag{3}$$

Definition 4 *A fuzzy measure μ is said to be k-*order additive *or simply k-*additive *if its Möbius transform $m(A) = 0$ for any A such that $|A| > k$, and there exists at least one subset A of X of exactly k elements such that $m(A) \neq 0$.*

Thus, k-additive measures can be represented by a limited set of coefficients, at most $\sum_{i=1}^{k} \binom{n}{i}$ coefficients.

3 Multicriteria decision making and game theory

As mentionned in the introduction, many ideas presented in this paper were inspired by cooperative game theory [29]. In this framework, X is a set of players, any subset $A \subset X$ is called a *coalition*, and μ, called the *characteristic function of the game*, expresses the *worth* (i.e. the amount of money the coalition will earn if the game is played) of any possible coalition. In general, μ is not supposed to be monotonic with respect to inclusion, and may take negative values.

A central problem in game theory is around the concept of *value*. It stems from the following problem: let $\mu(X)$ be the total worth of the game. Knowing the worth of each coalition, that is, groups of players, what is the monetary value of a single player? Obviously, this is not $\mu(\{i\})$, since it may happen that $\mu(\{i\}) = 0$, while every time player i joins a coalition $K \subset X \setminus \{i\}$, the worth of the new coalition is considerably greater. This means that i has been very useful, and should be rewarded by an amount $v_i > 0$. The value of the game is then the vector $[v_1 \cdots v_n]$. Shapley has proposed the following definition, now called the *Shapley value* (while a given v_i is the *Shapley index*).

Definition 5 *Let μ be a fuzzy measure on X. The* Shapley index *for every $i \in X$ is defined by*

$$v_i := \sum_{K \subset X \setminus i} \frac{(n - |K| - 1)!|K|!}{n!} [\mu(K \cup \{i\}) - \mu(K)],$$

The Shapley value *of μ is the vector $v(\mu) = [v_1 \cdots v_n]$.*

The Shapley index v_i can be interpreted as a kind of average value of the contribution of player i alone in all coalitions. The Shapley value represents a true sharing of the total amount $\mu(X)$, since $\sum_{i=1}^{n} v_i = \mu(X)$, and is also a linear operator over the set of games.

The analogy with the multicriteria decision making can be done as follows. The worth of a coalition of criteria is the importance of the coalition, or better, its importance or power to make alone the decision (without the remaining criteria). Obviously $\mu(X)$ has the maximal value, being 1 by convention. Then the Shapley index v_i expresses the relative importance of a single criterion into the decision problem, i.e. to what degree i is necessary to be kept in the set of criteria.

4 Interaction among criteria

The Shapley importance index is not enough to have a good description of the behaviour of players (or criteria) in a game (decision problem). Obviously, the Shapley index v_i does not reduce to $\mu(\{i\})$ because players interact together: either they have interest to cooperate, or not. Strangely enough, to our knowledge, there is no notion of interaction in game theory, perhaps because this is not the core concern of game theory. In multicriteria decision making however, the situation is different, and the comprehension of interaction phenomena between criteria is most informative. Let us explain on a simple example that importance of criteria alone is not sufficient to describe a decision model.

Let us consider two criteria 1 and 2, and four alternatives A, B, C, D to rank, as represented on figure 1. We suppose the criteria equally impor-

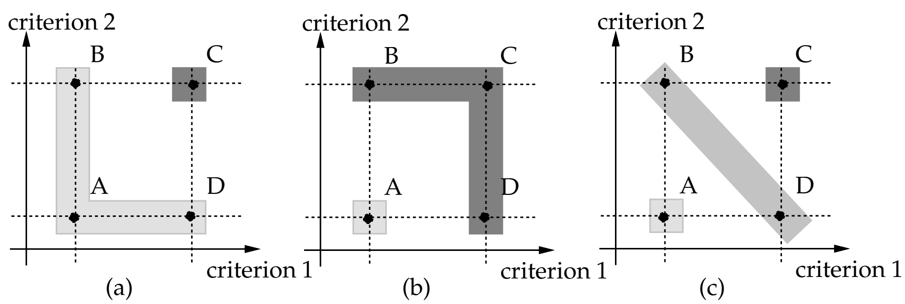

Figure 1: Different cases of interaction

tant for making decision. On the axes we represent the grade of satisfaction (score) of the alternatives for each criterion. As we normally prefer alternatives satisfying as much as possible criteria, every decision maker will have a strict preference for C over A. The case of alternatives B and D is more delicate. One decision maker may consider that B and D are equally bad as A, since neither of them satisfy both criteria, but only a single one. We could say that for this decision maker, criteria act conjunctively, so that both of them have to be satisfied. This is the case of figure 1(a), and we call this a case of *positive interaction* or *positive synergy* between criteria: although the importance of a single criterion for the decision is almost zero, the importance of the pair is large. The criteria can be said to be *complementary*.

Another decision maker may consider that B and D are equally good as C. In this case, criteria are considered to act disjunctively, and it is sufficient that one of them is satisfied (fig. 1(b)). Here we speak of *negative interaction* or *negative synergy*: the union of criteria does not bring anything, and the importance of the pair is almost the same as the importance of the single criteria. They are said to be *redundant*.

The third case (fig. 1(c)) is intermediary. Here the decision maker thinks that because B and D satisfy one criterion, they are better than A, but worse than C which satisfies both of them. In other words, the importance of the pair is more or less the sum of the individual importances of criteria: they act *independently* and there is no interaction between them.

The basic quantity for defining interaction seems then to be $\mu(\{i,j\}) - \mu(\{i\}) - \mu(\{j\})$. But as it was the case for the definition of the importance index, one has to examine what happens when i, j, and $\{i,j\}$ are added to coalitions, so that the basic quantity becomes $\mu(K \cup \{i,j\}) - \mu(K \cup \{i\}) - \mu(K \cup \{j\}) + \mu(K)$. Murofushi and Soneda [24], based on considerations of multiattribute utility theory [19], arrived at the following definition which exactly reflects the above discussion.

Definition 6 *Let μ be a fuzzy measure on X. The* interaction index *of elements $i, j \in X$ is defined by*

$$I_{ij} := \sum_{K \subset X \setminus \{i,j\}} \frac{(n - |K| - 2)!|K|!}{(n-1)!} [\mu(K \cup \{i,j\}) - \mu(K \cup \{i\}) - \mu(K \cup \{j\}) + \mu(K)], \tag{4}$$

The definition can be enlarged to any coalition, as done by Grabisch [9].

Definition 7 *Let μ be a fuzzy measure on X. The* interaction index *for any coalition $A \subset X$ is defined by*

$$I(A) := \sum_{B \subset X \setminus A} \frac{(n - |B| - |A|)!|B|!}{(n - |A| + 1)!} \sum_{C \subset A} (-1)^{|A \setminus C|} \mu(C \cup B). \tag{5}$$

It is clear that this is a generalization of both the Shapley value and the interaction index of Murofushi and Soneda, since v_i coincides with $I(\{i\})$ and I_{ij} with $I(\{i,j\})$.

An axiomatization of I has been built by Grabisch and Roubens [16]. It is seen that one of the fundamental axioms is the following:

> *Dummy axiom:* If i is a dummy player, then for every $A \subset X \setminus \{i\}$, $A \neq \emptyset$, $I(A \cup \{i\}) = 0$.

We say that i is a dummy player (criterion) if $\mu(A \cup \{i\}) = \mu(A) + \mu(\{i\})$ for any $A \subset X \setminus \{i\}$. This axiom expresses the very meaning of interaction. A dummy player brings only its worth to the coalition, nothing more, nothing less. This means that he/she has no interaction with any coalition. This axiom together with linearity over games is sufficient to get the characteristic form of the alternating sum in (4) and (5).

Relating the interaction index with k-additivity, we just mention the following fundamental property:

Property 1 *Let μ be a k-additive measure on X. Then*

(i) $I(A) = 0$ *for every $A \subset X$ such that $|A| > k$,*

(ii) $I(A) = m(A)$ *for every $A \subset X$ such that $|A| = k$,*

5 The Choquet integral for 2-additive measures

The case of 2-additive measure is particularly interesting. It remains simple (only quadratic complexity) and allows the modelling of interaction. It is possible to express the Choquet integral in the case of 2-additive measures, by using only the interaction index, as follows. Let $t_1, \dots, t_n$ be scores on criteria.

$$\mathcal{C}_\mu(t_1, \dots, t_n) = \sum_{I_{ij}>0} (t_i \wedge t_j) I_{ij} + \sum_{I_{ij}<0} (t_i \vee t_j)|I_{ij}| + \sum_{i=1}^{n} t_i (v_i - \frac{1}{2} \sum_{j \neq i} |I_{ij}|), \quad (6)$$

with the property that $v_i - \frac{1}{2}\sum_{j\neq i} |I_{ij}| \geq 0$ for all i. Here $|I_{ij}|$ denotes the absolute value of I_{ij}. It can be seen that the Choquet integral for 2-additive measures can be decomposed in a conjunctive, a disjunctive and an additive part, corresponding respectively to positive interaction indices, negative interaction indices, and the Shapley value. This makes clear the precise meaning of I_{ij} in the framework of the Choquet integral:

- a positive I_{ij} implies a conjunctive behaviour between i and j. This means that the simultaneous satisfaction of criteria i and j is significant for the global score, but a unilateral satisfaction has no effect.

- a negative I_{ij} implies a disjunctive behaviour, which means that the satisfaction of either i or j is sufficient to have a significant effect on the global score.

- the Shapley value acts as a weight vector in a weighted arithmetic mean. This represents the linear part of Choquet integral. It will be small if interaction indices are large.

6 Veto and favor

(see [10] for details)

Definition 8 *Suppose $\mathcal{H}$ is an aggregation operator being used for a multicriteria decision making problem. A criterion i is a* veto *for $\mathcal{H}$ if for any n-uple $(t_1, \dots, t_n) \in \mathbb{R}^{+n}$ of scores,*

$$\mathcal{H}(t_1, \dots, t_n) \leq t_i.$$

Similarly, criterion i is a favor *for $\mathcal{H}$ if for any n-uple $(t_1, \dots, t_n)$ of scores,*

$$\mathcal{H}(t_1, \dots, t_n) \geq t_i.$$

This means that when criterion i is a veto, if the score on i is high, it has no effect on the evaluation, but if it is low, the global score will be low too, whatever the values of the other scores are. The concepts of veto and favor have been already proposed by Dubois and Koning in the context of social choice functions,[3] where "favor" was called "dictator".

Fuzzy measure can represent veto and favors, as shown in the following proposition.

Property 2 *For the Choquet integral, i is a veto if and only if the fuzzy measure satisfies $\mu(A) = 0$ whenever $i \notin A$. Such fuzzy measures are denoted $\mu^{i\wedge}$. Similarly, i is a favor if and only if the fuzzy measure satisfies $\mu(A) = 1$ whenever $i \in A$. Such fuzzy measures are denoted $\mu^{i\vee}$.*

Remark that if criterion i is both a veto and a favor, then it is a dictator, i.e. $\mathcal{C}_\mu(t_1, \dots, t_n) = t_i$. Another consequence of the definition is that for a given $\mathcal{H}$, it is not possible to have simultaneously a veto on i and a favor on j, $j \neq i$ since having $\mathcal{H}(t_1, \dots, t_n) \leq t_i$ and $\mathcal{H}(t_1, \dots, t_n) \geq t_j$ is not compatible in general.

Let us examine the interaction representation of $\mu^{i\wedge}$ and $\mu^{i\vee}$. Denoting $I^{i\wedge}_{jk}$ and $I^{i\vee}_{jk}$ their respective interaction indices, it is easy to show that

$$I^{i\wedge}_{ik} \geq 0, \quad I^{i\vee}_{ik} \leq 0, \;\; \forall k \neq i. \tag{7}$$

Property 2 and equation (7) show that if i is a veto, then *necessarily* $I_{ik} \geq 0$, for any $k \neq i$ (similarly for a favor), but this is not a sufficient condition. Simple results can be given for 2-additive measures. The following can be shown.

Property 3 *Let μ be a 2-additive measure. Criterion i is a veto for the Choquet integral if and only if the following conditions are satisfied*

(i) $I_{ik} \geq 0, \quad \forall k \neq i,$

(ii) $I_{kl} = 0, \quad \forall k, l \neq i,$

(iii) $v_k = \dfrac{1}{2} I_{ik}, \quad \forall k \neq i.$

Similarly, i is a favor if and only if *(i)* $I_{ik} \leq 0, \quad \forall k \neq i,$

(ii) $I_{kl} = 0, \quad \forall k, l \neq i,$

(iii) $v_k = -\frac{1}{2} I_{ik}, \quad \forall k \neq i.$

It is possible to generalize the concept of veto to several criteria as follows. A group $A \subset X$ of criteria is a veto (resp. a favor) for $\mathcal{H}$ if every criterion in A is a veto (resp. a favor). This leads to the following equation in the case of a veto

$$\mathcal{H}(t_1, \ldots, t_n) \leq \bigwedge_{i \in A} t_i, \tag{8}$$

and in the case of a favor

$$\mathcal{H}(t_1, \ldots, t_n) \geq \bigvee_{i \in A} t_i. \tag{9}$$

For the Choquet integral, property 2 generalizes easily: a veto effect on a coalition A of criteria is obtained if and only if the fuzzy measure satisfies $\mu(B) = 0$ whenever $B \not\supset A$. The interaction of such measures satisfies $I(A \cup \{k\}) \geq 0, \forall k \notin A$. Similarly, a favor effect is obtained for A if and only if the fuzzy measure satisfies $\mu(B) = 1$ whenever $B \cap A \neq \emptyset$, and the interaction of such measure satisfies $I^{A\vee}(A \cup \{k\}) \leq 0, \forall k \notin A$.

7 Practical identification of a fuzzy measure

We present in this section two methods of identifying the fuzzy measure based on experimental data, i.e. examples given by a decision maker. As the reader will see, the two methods do not use exactly the same kind of input: the first one (based on minimization of squared error) needs (numerical) scores on criteria *and* the (numerical) global score, while the second one (based on constraint satisfaction) does not need a global score but only a ranking of the acts or objects to be evaluated, however it needs in addition some indication on the importance and interaction of the criteria.

7.1 Minimization of squared error

The idea of minimizing a squared error criterion to identify the fuzzy measure in a Choquet integral model has been advocated by several authors, essentially in Japan. We should cite in particular Tanaka and Sugeno [33, 34] in an application of printed color images evaluation, Mori and Murofushi [22], and Nakamori [27] in environmental evaluation.

Considering l acts or objects to be evaluated, we suppose that the decision maker is able to assess a numerical score for each act and each criterion, and also a numerical global score for each act. For act number k, we denote by $z_{k1}, \ldots, z_{kn}$ the scores on each criterion, and by y_k the global score. We want to find the fuzzy measure μ which minimizes the total squared error of the model, i.e.

$$E^2 = \sum_{k=1}^{l} (\mathcal{C}_\mu(z_{k1}, \ldots, z_{kn}) - y_k)^2 \tag{10}$$

under the constraint of monotonicity of the fuzzy measure. This can be put under a quadratic program of the form (see [13])

minimize $\frac{1}{2}\mathbf{u}^t\mathbf{D}\mathbf{u} + \mathbf{c}^t\mathbf{u}$
under the constraint $\mathbf{A}\mathbf{u} + \mathbf{b} \geq \mathbf{0}$

where $\mathbf{u}$ is a (2^n-2) dimensional vector containing all the coefficients of the fuzzy measure μ (except $\mu(\emptyset)$ and $\mu(X)$ which are fixed), $\mathbf{D}$ is a (2^n-2) dimensional square matrix, $\mathbf{c}$ a (2^n-2) dimensional vector, $\mathbf{A}$ a $n(2^{n-1}-1) \times (2^n-2)$ matrix, and $\mathbf{b}$ a $n(2^{n-1}-1)$ dimensional vector. The solution is in general not unique (see [21] for a study of this question), and it is possible to translate the quadratic program expressed in terms of μ to another quadratic program expressed in terms of the Möbius transform m, or directly the interaction index I (again see [21]). This allows to deal easily with k-additive measures, and to add constraints on interaction index values given by the decision maker.

The program can be solved by any standard method of quadratic optimization, although matrix $\mathbf{D}$ may be ill-conditioned (rank $< 2^n-2$). In the sequel, we have used the method of Powell-Schittkowski, referred hereafter as the *optimal quadratic method*.

Experiments on real data have shown some drawbacks of this method.

- if there is too few data[1], the solution is not unique of course, and the solution proposed by the program may be counterintuitive, because many coefficients are near 0 or 1.
- as n grows up, the dimensions of vectors and matrices grows exponentially, so does the memory required and the computation time. $n=8$ is already a large value, and $n=10$ is nearly infeasible.

For these reasons, some authors have looked for more heuristic methods, as Ishii and Sugeno [18] and Mori and Murofushi [22]. Based on this last one, Grabisch has proposed an optimization algorithm [7], which although suboptimal, gives better results than previous attempts. It is referred hereafter as the *heuristic least mean squares algorithm (HLMS)*. The basic idea is that, in

[1]Let us say much less than $n!/[(n/2)!]^2$, as proposed in [14]. In fact, there is not yet any definitive result on the minimum number of data for a correct identification, where "correct" is related to the number of solutions of the quadratic program. In [21], Miranda and Grabisch have given counterintuitive examples where the number of necessary data is 2 (!) whatever the value of n, and where $n!$ data (which is greater than 2^n-2, the number of variables, if $n>3$) is not sufficient for having a unique solution.

the absence of any information, the most non arbitrary (least specific) way of aggregation is the arithmetic mean (provided scores are on a difference scale), thus a Choquet integral with respect to an additive equidistributed fuzzy measure. Any input of information tends to move away the fuzzy measure from this equilibrium point. This means that, in case of few data, coefficients of the fuzzy measure which are not concerned with the data are kept as near as possible to the equilibrium point, in order to ensure monotonicity. Thus, in this algorithm, there is no problem of having too few data.

Experiments done in classification problems show the good performance of the algorithm, even better than the optimal method when n is large. Especially, the memory and computation time required are much smaller than for the quadratic program, and it is possible to treat problems with $n = 16$.

This method has been applied to a practical case, namely the evaluation of cosmetics [12].

7.2 Constraint satisfaction

As indicated at the beginning of this section, we suppose now that we have an expert who is able to tell the relative importance of criteria and the kind of interaction between them, if any.

Formally, keeping previous notations, the input data of the problem can be summarized as follows.

- The reference set of objects $A : \{1, \ldots, k, \ldots, \ell\}$, and the set of criteria $X : \{1, \ldots, i, \ldots, n\}$.
- A table of individual scores (performances) : $\{z_{ki},\ k \in A,\ i \in X\}$.
- A partial preorder $\geq_A$ on A (partial ranking of the objects on a global basis).
- A partial preorder $\geq_X$ on X (partial ranking of the criteria).
- A partial preorder $\geq_P$ on the set of pair of criteria (partial ranking of interaction indices).
- The sign of interaction between some pairs of criteria translating the synergy, independence or redundancy between these pairs.

The global scores are not needed in this approach.

All this information can be translated in terms of linear equalities or inequalities linking the unknown "weights" μ (constraint satisfaction problem).

Marichal and Roubens [20] proposed to solve the following linear program :

$$\max z = \varepsilon$$

subject to

$\varepsilon > 0$ (positive slack variable)

$$\left.\begin{array}{l} C_\mu(k) - C_\mu(k') \geq \delta + \varepsilon \text{ if } k >_A k' \\ -\delta \leq C_\mu(k) - C_\mu(k') \leq \delta \text{ if } k \sim_A k' \end{array}\right\} \begin{array}{c}\text{partial semiorder on } A \\ \text{with threshold } \delta\end{array}$$

$$\left.\begin{array}{l} \mu(\{i\}) - \mu(\{j\}) \geq \varepsilon \text{ if } i >_X j \\ \mu(\{i\}) = \mu(\{j\}) \quad \text{if } i \sim_X j \end{array}\right\} \text{ranking of criteria}$$

$$\left.\begin{array}{l} I_{ij} - I_{pq} \geq \varepsilon \text{ if } \{i,j\} >_P \{p,q\} \\ I_{ij} = I_{pq} \quad \text{if } \{i,j\} \sim_P \{p,q\} \end{array}\right\} \text{ranking of pairs of criteria}$$

$$\left.\begin{array}{l} I_{ij} \geq \varepsilon \ \text{ (resp. } \leq -\varepsilon) \ \text{ if } sign\ I_{ij} = +1 \text{ (resp. } sign\ I_{ij} = -1) \\ I_{ij} = 0 \ \text{ otherwise} \end{array}\right\} \begin{array}{c}\text{sign of some} \\ \text{interactions}\end{array}$$

$\sum_{i=1}^{n} v_i = \sum_{i=1}^{n} I(\{i\}) = 1$ (boundary condition on the values)

$\mu(A) \leq \mu(B)$ for all $A \subset B \subset X$ (monotonicity conditions).

$C_\mu(k) = C_\mu(z_{k1}, \ldots, z_{k\ell})$ represents the unknown global score for object k and δ represents the threshold level that should be reached by the difference between global scores to consider that one object should be significantly preferred to another object.

All the previous expressions can be rewritten in terms of the Möbius transform m related to μ. The fact that we suppose by reason of simplicity the fuzzy measure μ to be 2-additive gives $m(S) = 0$ for every S such that $|S| > 2$ and there is no distinction between the Shapley or Banzhaf values related to the criteria.

It has been proved (see [11]) that in the 2-additive case :

$$\begin{aligned} C_\mu(k) &= \sum_{i=1}^{n} m(\{i\}) z_{ki} + \sum_{\{i,j\} \subset X} m(\{i,j\})[z_{ki} \wedge z_{kj}] \\ I(\{i\}) &= v_i = m(\{i\}) + \frac{1}{2} \sum_{j \subset N \setminus i} m(\{i,j\}) \\ I(\{i,j\}) &= m(\{i,j\}) \\ \mu(\{i\}) &= m(\{i\}). \end{aligned}$$

Moreover, the monotonicity conditions are equivalent to (see [11])

$$\begin{aligned} m(\{i\}) &\geq 0 && \text{for all } i \in X \\ m(\{i\}) + m(\{i,j\}) &\geq 0 && \text{for all } i,j \in X \\ m(\{i\}) + m(\{i,j\}) + m(\{i,k\}) &\geq 0 && \text{for all } i,j,k \in X. \end{aligned}$$

The objective function that is chosen (maximize the value of the positive slack variable) to solve a linear program is justified by the following result (see [28]) :

$x \in \mathbb{R}^s$ is a solution of the linear system (constraint satisfaction problem)

$$\begin{cases} \sum_{j=1}^{s} a_{ij} x_j \leq b_i, & i = 1, \ldots, p \\ \sum_{j=1}^{s} c_{ij} x_j < d_i, & i = 1, \ldots, q \end{cases}$$

if and only if the following linear program

$$\max z = \varepsilon$$

subject to

$$\begin{cases} \sum_{j=1}^{s} a_{ij} x_j \leq b_i, & i = 1, \ldots, p \\ \sum_{j=1}^{s} c_{ij} x_j \leq d_i - \varepsilon, & i = 1, \ldots, q \end{cases}$$

has an optimal value $x^* \in \mathbb{R}^s$ with an optimal value $\varepsilon^* > 0$. In this case, x^* is a solution of the constrained satisfaction problem.

8 Illustrative example

In this section, we illustrate on a fictitious example the two approaches explained above for the determination of the fuzzy measure.

8.1 Definition of the example

We consider the problem of the evaluation of trainees learning to drive military vehicles. Trainees are evaluated by instructors according to 4 criteria:

C.1 **precision:** the trainees are supposed to fire on targets, as precisely as possible. The percentage of success during the exercice is computed.

C.2 **rapidity:** the trainees have to detect as fast as possible the target, in order to fire on it. The elapsed time between the appearance of the target and the detection is measured (in tu (time unit)).

C.3 **driving:** in order to go from one point to another, the trainee has to choose a suitable trajectory, or to follow a given one as precisely as possible. A qualitative score is given by the instructor, going from A (excellent) to E (hopeless).

C.4 **communication:** the trainee is supposed to belong to some unit, and thus he has to understand and obey orders, and also to report actions. As for the driving criterion, a qualitative score is given by the instructor, going from A (perfect) to E (incontrollable).

We consider 5 trainees, whose names (they may be considered as a modern version of knights...) and performances on each criterion are given in table 1. The instructor can make the following comments about the scores on the criteria:

name	precision (%)	rapidity (tu)	driving	communication
Arthur	90	2	B	D
Lancelot	80	4	B	B
Yvain	95	5	C	A
Perceval	60	6	B	B
Erec	65	2	C	B

Table 1: Performances of the different trainees

- C.1 (precision): over 90% of success is perfect, below 50% is totally unacceptable.
- C.2 (rapidity): below 2 tu is perfect, over 10 tu is totally unacceptable.
- criteria C.3 and C.4 are already expressed under the form of a score.

This permits us to draw utility curves in order to derive numerical scores (degrees of satisfaction). They are given on figure 2. Applying these utility functions gives the following numerical scores for the trainees (table 2).

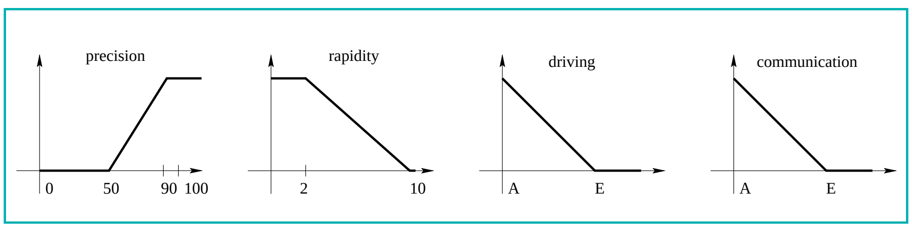

Figure 2: Scores on the different criteria

Looking at the performances of the different trainees, the instructor is able to rank the trainees, as given in table 3. There are three predetermined classes, called good, average, bad. In each class, a ranking is done, labelling by 1 the best in the class, by 2 the second best, etc. According to this notation,

name	precision	rapidity	driving	communication
Arthur	1.000	1.000	0.750	0.250
Lancelot	0.750	0.750	0.750	0.750
Yvain	1.000	0.625	0.500	1.000
Perceval	0.250	0.500	0.750	0.750
Erec	0.375	1.000	0.500	0.750

Table 2: Numerical scores on criteria

name	class	rank in the class
Arthur	bad	2
Lancelot	good	1
Yvain	good	2
Perceval	bad	1
Erec	average	1

Table 3: Ranking of the 5 trainees

Arthur is the worst driver, and Lancelot the best one. The instructor is able to comment about his decision as follows.

- Arthur is excellent at firing on targets: he is both very precise and very quick. He also drives well, but he is very bad at communication. This is rather dangerous, since he may not obey orders, nor report on what he is doing. Despite his technical ability, he is completely unsuitable for driving military vehicles, and should be put in the bad class.

- Lancelot is not as good as Arthur for firing, but indeed sufficiently good. Since he has no weak point and has a satisfying score on all criteria, he has to be put without any doubt in the good class.

- Yvain is the best at precision (even better than Arthur), but he is somewhat weak at rapidity. To a certain extent, his deficiency on rapidity can be compensated by his outstanding score on precision, since once he detects the target he will not miss it. Also, Yvain is perfect for communication, and although only average on driving, he can be put into the good class, but behind Lancelot however.

- Perceval is weak at firing since he has a bad score on precision and rapidity too. Whatever the remaining score, this cannot be accepted, and we have to put him in the bad class. It is not clear whether he is worse or better than Arthur, but finally the instructor put him before.

- Erec has a bad precision but detects quite quickly. To a certain extent, this can compensate the bad precision. Since he is rather good on the other criteria, he can be put in the average class.

8.2 Approach by the minimization of the quadratic error

Here we need a numerical global score and not only a ranking. We use an arbitrary numerical mapping to transform this ranking into numerical global scores, in order to apply our approach. The mapping is based on contiguous intervals for each class, and an equirepartition in each class (i.e. a sample alone in a class will be attributed the mean of the interval, 2 samples in a class will divide the interval in 3 parts, etc.). The intervals for this application are given in table 4. This leads to table 5, which summarizes

class	interval for the global score
good	$[0.75, \quad 1.0]$
average	$[0.4, \quad 0.75]$
bad	$[0.0, \quad 0.4]$

Table 4: Mapping from class and rank to [0,1]

all the data. It can be noticed that the global evaluation seems to be too ex-

name	precision	rapidity	driving	communication	global
Arthur	1.000	1.000	0.750	0.250	0.133
Lancelot	0.750	0.750	0.750	0.750	0.917
Yvain	1.000	0.625	0.500	1.000	0.833
Perceval	0.250	0.500	0.750	0.750	0.267
Erec	0.375	1.000	0.500	0.750	0.575

Table 5: Numerical data on criteria and global performance (after conversion)

treme, putting a too low score for Arthur (below the minimum of scores on the criteria), and a too high for Lancelot (above the maximum of scores on the criteria). We perform however the two available optimization methods described above. In what follows, the Shapley value will be always normalized (i.e. multiplied by n, the number of criteria), so that a value of 1 indicate an average importance for the criterion.

We begin by the heuristic least mean square algorithm (HLMS), with parameters $\alpha = \beta = 0.05$, and 300 iterations[2]. The result is listed below.

```
fuzzy measure coefficients
```

[2]The parameter α is the learning rate in the gradient descent algorithm, while β is a coefficient used for updating (see [7] for details).

```
-------------------------
subset  1  : 0.16929
subset  2  : 0.175353
subset  3  : -9.11165e-08
subset  4  : -4.55582e-08
subset  1   2  : 0.175396
subset  1   3  : 0.175396
subset  1   4  : 0.613114
subset  2   3  : 0.175396
subset  2   4  : 0.425354
subset  3   4  : -4.55582e-08
subset  1   2   3  : 0.175396
subset  1   2   4  : 0.787705
subset  1   3   4  : 0.739775
subset  2   3   4  : 0.425375
subset  1   2   3   4  : 1
Arthur :   model output  = 0.381547, desired output = 0.133333
Lancelot : model output  = 0.750000, desired output = 0.916667
Yvain :    model output  = 0.828381, desired output = 0.833333
Perceval : model output  = 0.356344, desired output = 0.266667
Erec :     model output  = 0.578349, desired output = 0.575000
Results
-------
criterion value : 0.097465
mean error      : 0.139618
best approximation for   Erec (|error| = 0.003349)
worst approximation for  Arthur (|error| = 0.248213)

Shapley index (normalized)
-------------------------
precision     : 1.398027
rapidity      : 0.861737
driving       : 0.280457
communication : 1.555321

Matrix of interaction index
---------------------------
I( 1, 2)=-0.190  I( 1, 3)= 0.086  I( 1, 4)= 0.486
                 I( 2, 3)= 0.020  I( 2, 4)= 0.232
                                  I( 3, 4)= 0.083
```

We can make the following comments.

- the most important criterion is communication, and then precision. driving seems to be negligible. Referring to the comments of the instructor, we see that there is no special mention of the driving criterion (so it does not take part in the decision), and that communication is very important.
- there is a negative interaction between precision and rapidity. This is

also in accordance with the instructor's comment, since a kind of compensation exists between these two criteria.

- there is a strong positive interaction between precision and communication, and rapidity and communication, meaning that a good driver *must* be good at precision *and* communication. In fact, communication is a (non strict) veto. This clearly reflects the instructor way of thinking, since he discarded Arthur because of the communication. The ordering given by the instructor is not satisfied, since we obtain:

  Yvain > Lancelot > Erec > Arthur > Perceval

  (two inversions) But observe that the two inverted trainees have nevertheless close global scores.

We apply now the optimal quadratic method. The results are as follows.

```
Fuzzy measure coefficients
--------------------------
subset  1  : 1e-06
subset  2  : 1e-06
subset  3  : 1e-06
subset  4  : 1e-06
subset  1   2  : 1e-06
subset  1   3  : 1e-06
subset  1   4  : 0.666667
subset  2   3  : 1e-06
subset  2   4  : 0.389743
subset  3   4  : 1e-06
subset  1   2   3  : 1e-06
subset  1   2   4  : 0.666667
subset  1   3   4  : 0.666667
subset  2   3   4  : 0.389743
subset  1   2   3   4  : 1
Arthur   : model output  = 0.250001, desired output = 0.133333
Lancelot : model output  = 0.750000, desired output = 0.916667
Yvain    : model output  = 0.833333, desired output = 0.833333
Perceval : model output  = 0.347436, desired output = 0.266667
Erec     : model output  = 0.521154, desired output = 0.575000
Results
-------
criterion value : 0.050812
mean error      : 0.100809
best approximation for  Yvain (|erreur| = 0.000000)
worst approximation for Lancelot (|erreur| = 0.166667)

Shapley index (normalized)
--------------------------
precision     : 1.147010
```

```
rapidity      : 0.593162
driving       : 0.333334
communication : 1.926493

Matrix of interaction index
---------------------------
I( 1, 2)=-0.084  I( 1, 3)= 0.111  I( 1, 4)= 0.583
                 I( 2, 3)= 0.111  I( 2, 4)= 0.306
                                  I( 3, 4)= 0.111
```

It can be seen that the model error is twice better, but the results are qualitatively the same (same ranking of the criteria, same signs for the interaction). However, in addition, communication has become a strict veto. The ranking is slightly better since only one inversion remains:

Yvain > Lancelot > Erec > Perceval > Arthur

We try now to modify the numerical global score, to be more in accordance with the instructor's thinking, and following the rule of being consistent with minimum and maximum. We obtain the following new table (table 6), after, let us say, some discussion with the instructor.

name	precision	rapidity	driving	communication	global
Arthur	1.000	1.000	0.750	0.250	0.3
Lancelot	0.750	0.750	0.750	0.750	0.75
Yvain	1.000	0.625	0.500	1.000	0.7
Perceval	0.250	0.500	0.750	0.750	0.35
Erec	0.375	1.000	0.500	0.750	0.5

Table 6: Numerical data on criteria and global performance (2nd version)

We obtain the following result (presented for the optimal method only, skipping details).

```
....
Results
-------
criterion value : 0.000000
....

Shapley index (normalized)
--------------------------
precision     : 0.977778
rapidity      : 0.888888
driving       : 0.637038
communication : 1.496296
```

```
Matrix of interaction index
---------------------------------
I( 1, 2)= 0.093  I( 1, 3)= 0.181  I( 1, 4)= 0.437
                 I( 2, 3)= 0.181  I( 2, 4)= 0.259
                                  I( 3, 4)= 0.237
```

It is remarkable that the system has succeeded to find an exact model (no error). Although the result shows the same ordering of values for Shapley and interaction indices, there are some significant differences with the previous case:

- the most important criterion is still communication, and the less important is still driving, but the importances are less contrasted. In particular, driving is more important than before, and is close to rapidity.
- there is still a strong positive interaction between communication and precision, but now there is no more compensation between precision and rapidity.

We can explain this as follows. If we want to maintain Yvain below Lancelot, i.e. lower than 0.75, it means that, since he is *perfect* on precision and communication, we must take into account his mediocre score on driving, —and thus putting some importance on this criteria—, and on rapidity, —which inevitably cancels the (supposed) compensativity between precision and rapidity.

We can verify this by modifying the data as follows. We suppose that, being aware of this fact, the instructor now changes his mind, and decides that Yvain is after all better than Lancelot, with a global score of 0.85 (others remain inchanged). The results are as follows.

```
....
Results
-------
criterion value : 0.000000
....

Shapley index (normalized)
--------------------------
precision     : 1.277779
rapidity      : 0.588888
driving       : 0.337038
communication : 1.796296

Matrix of interaction index
---------------------------------
I( 1, 2)=-0.007  I( 1, 3)= 0.081  I( 1, 4)= 0.637
                 I( 2, 3)= 0.081  I( 2, 4)= 0.159
                                  I( 3, 4)= 0.137
```

The effect of the modification is noticeable. driving has become much less important, and a slight effect of redundancy between precision and rapidity appears. However, it seems that there is a large difference in importance between precision and rapidity, in favor of the first. Looking at the scores of Erec, this is not surprising since he has got a mediocre global score, despite his performance on rapidity (perfect) and communication (good). In a last attempt, the instructor decides to raise a little the global score of Erec, from 0.5 to 0.6. The following results are obtained.

```
Arthur   : model output = 0.305785, desired output = 0.300000
Lancelot : model output = 0.750000, desired output = 0.750000
Yvain    : model output = 0.850000, desired output = 0.850000
Perceval : model output = 0.376033, desired output = 0.350000
Erec     : model output = 0.582645, desired output = 0.600000
Results
-------
criterion value : 0.001012
....

Shapley index (normalized)
--------------------------
precision     : 1.027825
rapidity      : 0.784847
driving       : 0.300001
communication : 1.887328

Matrix of interaction index
---------------------------------
I( 1, 2)=-0.152  I( 1, 3)= 0.100  I( 1, 4)= 0.548
                 I( 2, 3)= 0.100  I( 2, 4)= 0.278
                                  I( 3, 4)= 0.100
```

This time, a small residual error exists, but there is no inversion of the ranking. The results show as expected a noticeable gap between rapidity and driving, while precision and rapidity are now very close. Also, there is a strong redundancy effect between precision and rapidity. Remark also that we have become close to the first results, with the numerical mapping to convert classes and ranks into global scores, but with a much smaller error.

The results presented here shows the flexibility of the methodology, and its adequacy to reflect the decision strategy.

8.3 Approach based on constraint satisfaction

We consider the following data that is supposed to be given by the instructor.

- Table 2 as the table of z values (reproduced in Table 7)

- A total order on the reference set :

$$\text{Lancelot} > \text{Yvain} > \text{Erec} > \text{Perceval} > \text{Arthur}$$

- A partial order on X which corresponds to the Hasse diagram (for a definition see [28]) of figure 3. This partial order indicates that $\mu(4) >$

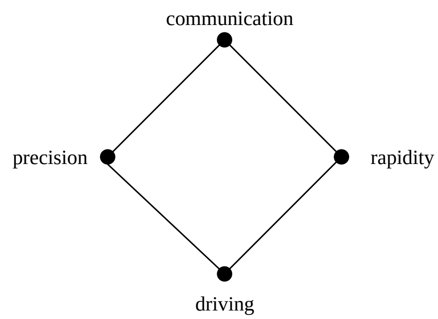

Figure 3: Partial order on the criteria

  $\mu(1)$, $\mu(4) > \mu(2)$, $\mu(1) > \mu(3)$, $\mu(2) > \mu(3)$ (and by transitivity : $\mu(4) > \mu(3)$).

  However nothing is said about the preference relation between precision and rapidity.

- Some information about the interactions :

$$I_{12} < 0,\ I_{14} > 0 \text{ and } I_{24} > 0.$$

  This information states that there is a negative interaction between precision and rapidity, a positive interaction between precision and communication and between rapidity and communication.

Note that, as expected, the data are not of the same type than for the former method.

Applying the linear programming method described in section 7.2 with δ equal to 0.05 (there should be at least a difference of 5% between the global scores of the trainees) we obtain the results of table 7. The (normalized) Shapley indices are given below, as well as the interaction indices (table 8).

Let us examine what happens in this example if we make a slight change in the data supplied by the decision maker, concerning the interaction. We have done two experiments.

- the instructor says there is *no interaction* between the criteria (thus all I_{ij} are 0). In this case, there is no solution, even with $\delta = 0$. This means that no solution can be obtained if one uses the classical weighted mean to solve this specific problem.

name	precision	rapidity	driving	communication	global
Arthur	1.000	1.000	0.750	0.250	0.39
Lancelot	0.750	0.750	0.750	0.750	0.75
Yvain	1.000	0.625	0.500	1.000	0.66
Perceval	0.250	0.500	0.750	0.750	0.48
Erec	0.375	1.000	0.500	0.750	0.57

Table 7: Numerical data on criteria and global performance

criterion	Shapley index
precision	0.80
rapidity	0.72
driving	0.88
communication	1.6

Interaction index		
I(1,2) = -0.04	I(1,3) = 0.23	I(1,4) = 0.13
	I(2,3)=0	I(2,4)) = 0.31
		I(3,4) = 0.21

Table 8: Shapley (left) and interaction (right) indices obtained by the constraint satisfaction method

- the instructror says that he is *unable to tell anything* about the interaction (thus there is no contraint on I_{ij}). In this case, the method gives exactly the same result as in table 8.

8.4 Comparisons

Strictly speaking, the methods are in fact not comparable, since they don't take exactly the same input, nor provide the same kind of output. To sum up :

- the MSE method (minimization of the squared error) needs only a global score, which can be provided (as in the example) by a ranking of the acts through a suitable mechanism (this last point is not so obvious, as shown in the example). The output is (apart the fuzzy measure) an estimation of the model error.

- the CS (constraint satisfaction) method needs only a ranking of the acts (and not a global score), but also a ranking on the importance of the criteria, and possibly some information on the interactions. There is no notion of model error in such approach : either there is a solution satisfying the constraints, or there is not.

The advantages and shortcomings of these two approaches can be summarized as follows.

- on the positive side, the MSE method gives always a solution, which fits at best[3] the given global scores. More importantly, the method does not need any information on the decision strategy (importance and interaction). It is perfectly suitable for *identifying* a hidden decision behaviour (of a consumer, etc.), as was done in [12].

  On the negative side, the method may disturb the ranking provided by the decision maker (as it was the case in the example).

- the CS method has the advantage of being based only on ordinal information, as it is often the case in real multicriteria decision problems. Also, the method does not violate the ranking provided by the decision maker. However, the method needs some information on the decision strategy in order to be efficient. Strictly speaking, one may use the method without such information (no constraint but the ranking of the acts), but then the space of feasible solutions may be so huge that the solution chosen has no real value in terms of decision strategy. This method is more suitable when one want to *define* or *build* a decision strategy in terms of importance and interaction.

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[3]optimal if the optimal quadratic method is used.

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