Chapter III:
Nuclear models

Summary

1. Introduction
2. Liquid drop model
3. Collective model
4. Fermi-gas model
5. Shell model

Introduction (1)

- Impossible to study exactly nuclei with more than 2 or 3 nucleons → resolution of the Schrödinger equation with too many variables →

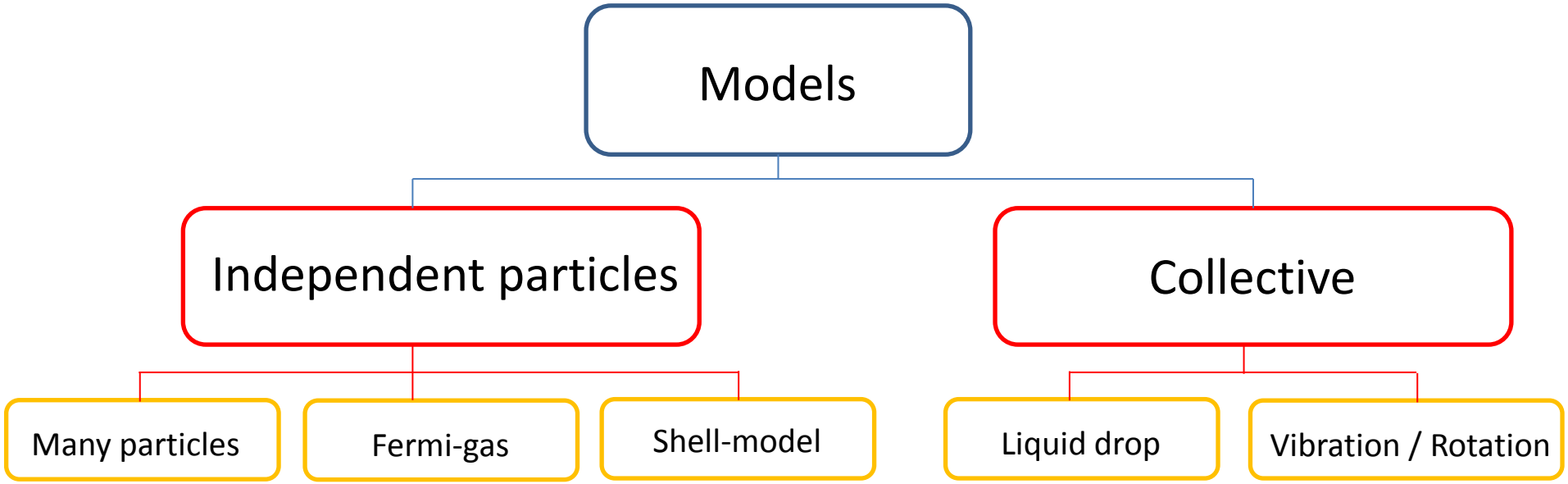
$$\left(\sum_{i=1}^{A} T_i + \sum_{j>i=1}^{A} V_{ij} + \dots \right) \Psi = E\Psi$$

- Even for 2 or 3 nucleons → study is imprecise → the exact nature of nuclear forces is not known (there exist 3-body interactions → no classical analog)
- Moreover relativistic effects are difficult to evaluate

Introduction (2)

- Approach for nuclei → a deliberately oversimplified approach is chosen → development of models → mathematically tractable and rich in physical insight
- Good model →
  - It must reasonably well account for previously measured nuclear properties
  - It must predict additional properties that can be measured in new experiments
- Attention → it exists no model capable to describe simultaneously all properties of the nucleus → each model has a limited range of applications

Introduction (3)



Individual properties of nucleons → nucleons are moving in a « mean » potential → they are independent

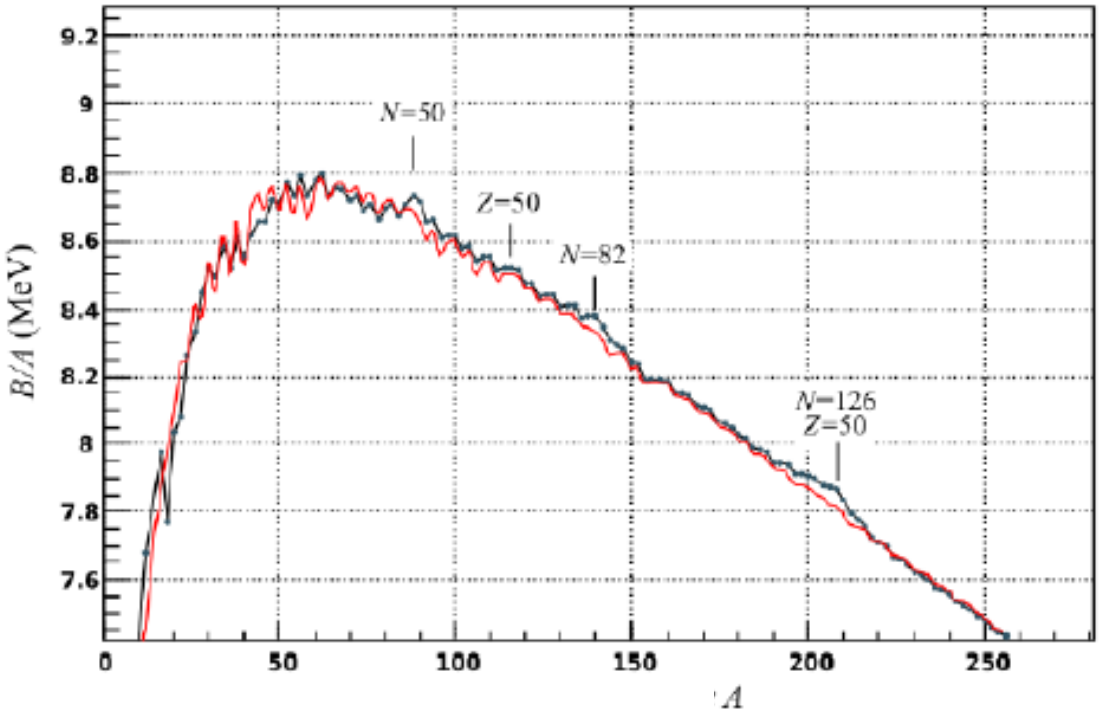
« Collective » properties of the nucleus → the nucleons strongly interact → properties are due to the nucleons in a whole

Liquid drop model (1)

- Liquid drop model developed by Gamow
- Nuclei are approximatively spherical → the volume if the sphere is $\propto A$ → each nucleon takes up an equal volume independently of the considered nucleus → the nuclear density is independent of A → the « nuclear material » is incompressible as a liquid drop
- The atomic nucleus behaves like the molecules in a drop of liquid
- Nuclear scale → the fluid is made of nucleons (protons and neutrons) which are held together by the strong nuclear force
- The nuclear forces on the nucleons on the surface are different from those on nucleons in the interior of the nucleus (the interior nucleons are completely surrounded by other attracting nucleons) → analogy with the forces that form a drop of liquid

Liquid drop model: applications (1)

- Important success of the liquid drop model → Bethe-Weizsäcker formula (also called semi-empirical mass formula – SEMF)



Comparison between result of SEMF and experiment

Liquid drop model: applications (2)

- Expression of the mass parabola → the more stable nucleus corresponds to the minimum of the parabola →

$$Z_{min} = \frac{[m_N - m(^1H)] + a_C A^{-1/3} + 4a_a}{2a_C A^{-1/3} + 8a_a A^{-1}} \simeq \frac{A}{2} \frac{1}{1 + 0.0078 A^{2/3}}$$

- Calculation of energy released during nuclear reaction →

$$Q = \sum_i (m_i c^2)_{initial} - \sum_j (m_j c^2)_{final}$$

- Study of spontaneous fission

Vibrational model (1)

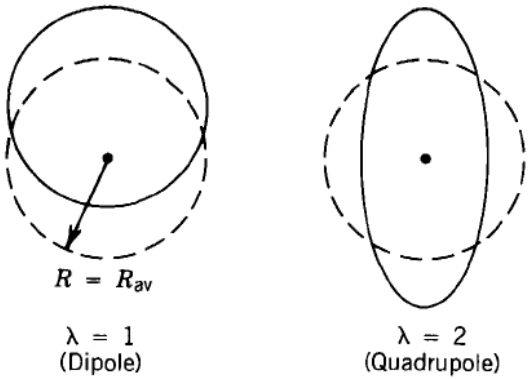
- Collective model → the nucleus is considered as a liquid drop (extension of the liquid drop model) vibrating at high frequency
- The average shape of the nucleus is spherical but the instantaneous shape is not
- Instantaneous coordinate *R*(*t*) of a point on the nuclear surface at (θ,ϕ) in terms of spherical harmonics →

$$R(t) = R_0 A^{1/3} + \sum_{\lambda \geq 1} \sum_{\mu=-\lambda}^{+\lambda} \alpha_{\lambda\mu}(t) Y_{\lambda\mu}(\theta, \phi)$$

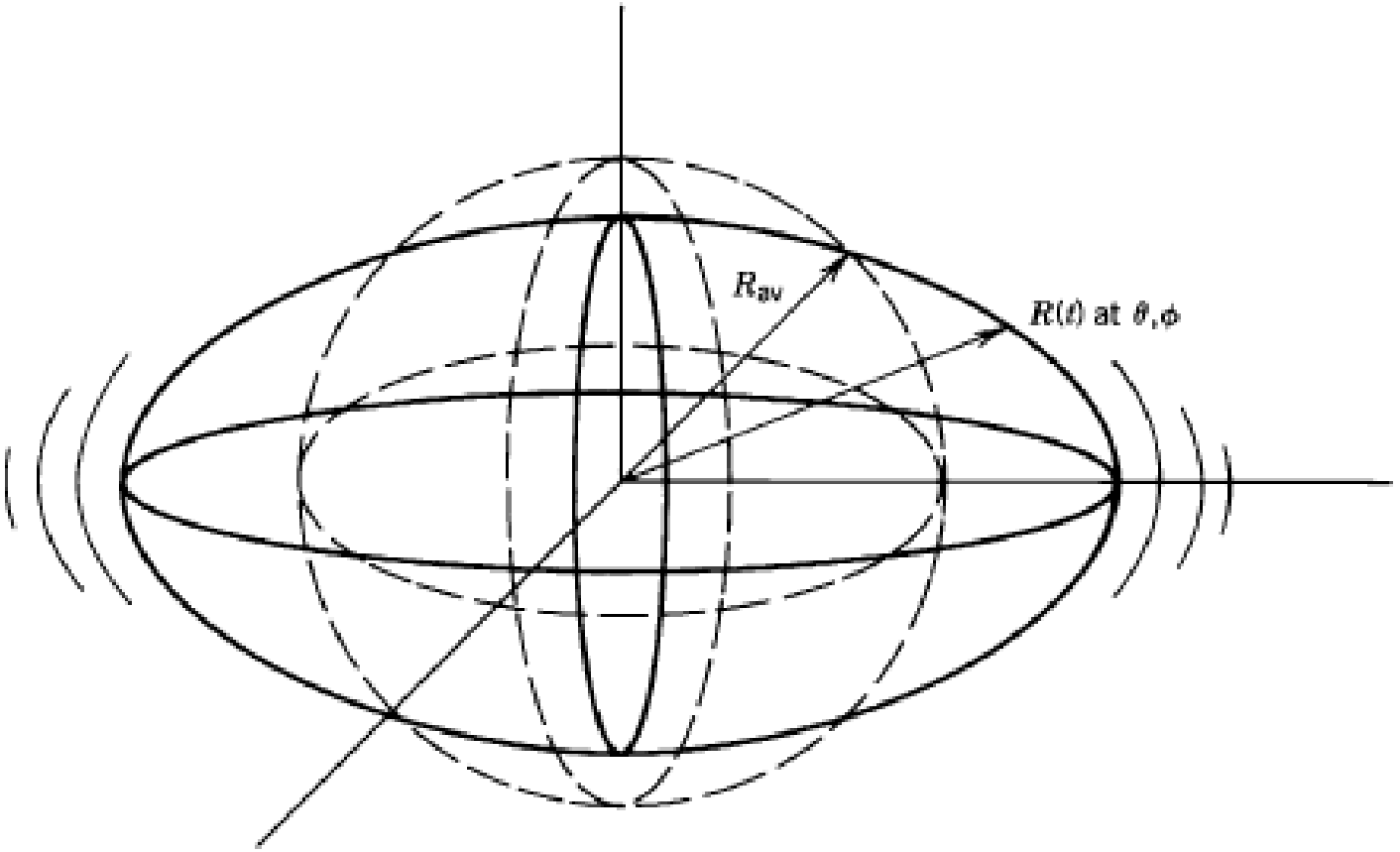
with $\alpha_{\lambda\mu}$ the amplitude of vibration

Vibrational model (2)

- λ = 0 corresponds to a variation of volume → impossible for incompressible fluid → monopolar mode is forbidden
- λ = 1 corresponds to a net displacement of the center of mass → cannot result from the action of internal nuclear forces → dipolar mode is not considered
- λ = 2 corresponds to the lowest mode to be considered → quadrupole mode → in analogy with quantum theory of electromagnetism (1 unit of electromagnetic energy is called a photon) → 1 quantum of vibrational energy is called a phonon



Vibrational model (3)



Vibrating nucleus with a spherical equilibrium shape

Vibrational model (4)

- When mechanical vibrations are produced → vibrational phonons are produced → 1 single unit of λ = 2 nuclear vibration is called *a quadrupole phonon*
- The vibration spectrum is composed of a ground state corresponding to the spherical nucleus, a first excited state corresponding to the energy of 1 phonon, a second state corresponding to 2 phonons,…
- Each state has degeneracy corresponding to different values of μ
- This model several predictions as the spectrum of ^{120}Te

Rotational model (1)

- Rotational motion can be observed only in nuclei with nonspherical equilibrium shapes
- These nuclei can have substantial distortions from spherical shape → they are called deformed nuclei
- They are only found in the mass ranges $150 < A < 190$ and $A > 220$ (rare earths and actinides)
- Common representation of the shape of these nuclei → ellipsoid of revolution
- This surface is described by (with β the deformation parameter) →

$$R(t) = R_0 A^{1/3}[1 + \beta Y_{20}(\theta, \phi)]$$

which is independent of ϕ → cylindrical symmetry

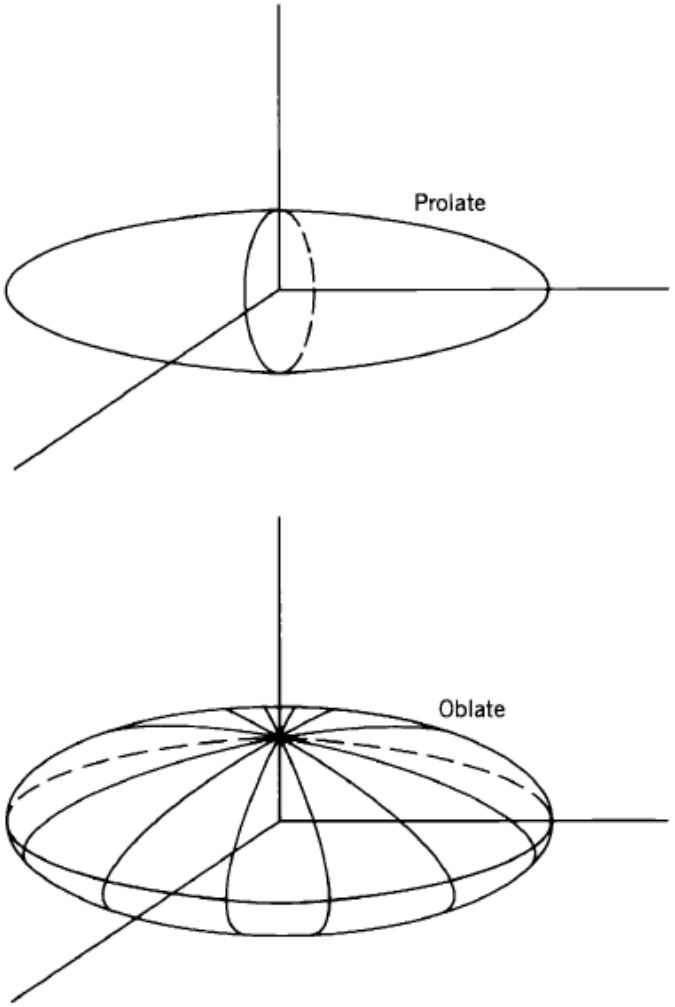
Rotational model (2)

- For $\beta > 0$ → the nucleus has the elongated form of a prolate ellipsoid
- For $\beta < 0$ → the nucleus has the flattened form of oblate ellipsoid
- β is related to the eccentricity of the ellipse as →

$$\beta = \frac{4}{3}\sqrt{\frac{\pi}{5}}\frac{\Delta R}{R_0 A^{1/3}}$$

- ΔR is the difference between the semimajor and semiminor axes of the ellipse
- Attention → previous expressions imply a volume ≠ $4/3\pi R_0^3 A$ → not quite exact approximation

Rotational model (3)



Rotational model (4)

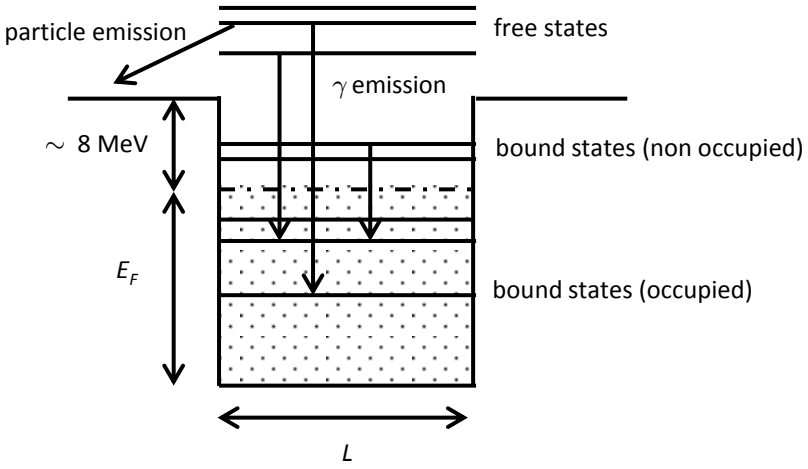
- Deformation of the nucleus → apparition of a large quadrupole moment Q_0
- A rotating object has kinetic energy → this kinetic energy is quantified via the angular momentum quantum number *I*
- Increasing *I* corresponds to adding rotational energy to the nucleus → the nuclear excited states form a sequence known as a rotational band (analogy with excited states in molecules → also presence of form rotational bands corresponding to rotations of the molecule about its center of mass)
- This model several predictions as the spectrum of ^{164}Er

Fermi-gas model: Principles

- The Fermi-gas model is a partly quantum model of independent particles
- It may be applied for large systems of weakly interacting fermions → particles follow Fermi-Dirac statistics leading to the Pauli exclusion principle
- Picture of the nucleus →
  - protons and neutrons are moving freely within the nuclear volume *V*
  - the interaction potential is generated by all nucleons
- Nucleons are in a potential well → in first approximation a rectangular potential is considered → constant inside the nucleus and stops sharply at its edge
- Neutrons and protons are distinguishable fermions → they are situated in two separate potential wells

Fermi-gas model: Potential well (1)

- Each energy state can be occupied by two nucleons with different spin projections
- All available energy states are filled by the pairs of nucleons → no free states → no transitions between the states
- The energy of the highest occupied state is the Fermi energy E_F
- The difference between the top of the well and the Fermi level is constant for most nuclei → it is just the average binding energy per nucleon B/A = 7–8 MeV



Fermi-gas model: Potential well (2)

- All nucleons are confined in a spherical box with volume *V*
- The Schrödinger equation for a particle in the box is →

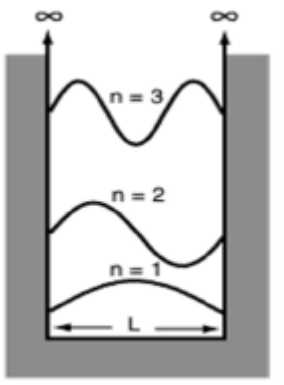
$$-\frac{\hbar^2}{2m}\Delta\psi(\boldsymbol{r}) = E\psi(\boldsymbol{r}) \text{ with } \psi(r_i) = 0 \text{ for } r_i = 0 \text{ or } L$$

- Considering

$$k^2 = \frac{2mE}{\hbar^2} \Rightarrow \psi(x,y,z) = N\sin(k_x x)\sin(k_y y)\sin(k_z z)$$

$$k_x = \frac{n_x\pi}{L}; k_y = \frac{n_y\pi}{L}; k_z = \frac{n_z\pi}{L}$$

- Each set (n_x,n_y,n_z) defines a solution



Fermi-gas model: Potential well (3)

- The nucleons have discrete energy states →

$$E = \frac{\hbar^2}{2m}k^2 = \frac{\hbar^2}{2m}[k_x^2 + k_y^2 + k_z^2]$$

- The Fermi energy is (p_F being the Fermi momentum) →

$$\frac{k_F^2 L^2}{\pi^2} = n_x^2 + n_y^2 + n_z^2 \Rightarrow E_F = \frac{\hbar^2}{2m}k_F^2 = \frac{p_F^2}{2m}$$

- The phase space is now divided in microscopic cells corresponding to the volume given by the incertitude principle of Heisenberg → the elementary volume occupied by a particle in this space phase is →

$$h^3 = (2\pi\hbar)^3$$

Fermi-gas model: Potential well (4)

- In the physical volume → the number of cells (= accessible states) *dn* containing a particle with momentum included in [*p*,*p* + *dp*] is

$$dn = \frac{d\boldsymbol{p}d\boldsymbol{r}}{(2\pi\hbar)^3}\frac{4\pi p^2 dpV}{(2\pi\hbar)^3}$$

- Integration up to p_F gives (taking into account the Pauli principle) →

$$n = 2 \times \frac{p_F^3 V}{6\pi^2\hbar^3} \Rightarrow p_F = \hbar\left(\frac{3\pi^2 n}{V}\right)^{1/3} \Rightarrow E_F = \frac{(\hbar c)^2}{2mc^2}\left(\frac{3\pi^2 n}{V}\right)^{2/3}$$

- Considering *n* = *Z* for protons / *n* = *N* for neutrons / V = $4/3\pi R_0^3 A$ →

$$Z = \frac{(p_F^p)^3 V}{3\pi^2\hbar^3} \text{ and } N = \frac{(p_F^n)^3 V}{3\pi^2\hbar^3}$$

Fermi-gas model: Potential well (5)

- Assuming that the proton and neutron potential wells have the same radius and $n = Z = N = A/2 \rightarrow p_F^p = p_F^n = p_F \rightarrow$

$$p_F = \left(\frac{9\pi}{8}\right)^{1/3} \frac{\hbar}{R_0} \quad \text{and} \quad E_F = \frac{1}{2mc^2}\left(\frac{9\pi}{8}\right)^{2/3}\left(\frac{\hbar c}{R_0}\right)^2$$

- Considering $mc^2 \approx 938$ MeV and $R_0 \approx 1.2$ fm and $\hbar c = 197$ MeV fm $\rightarrow$

$$p_F \approx 250 \text{ MeV/c} \quad \text{and} \quad E_F \approx 33 \text{ MeV}$$

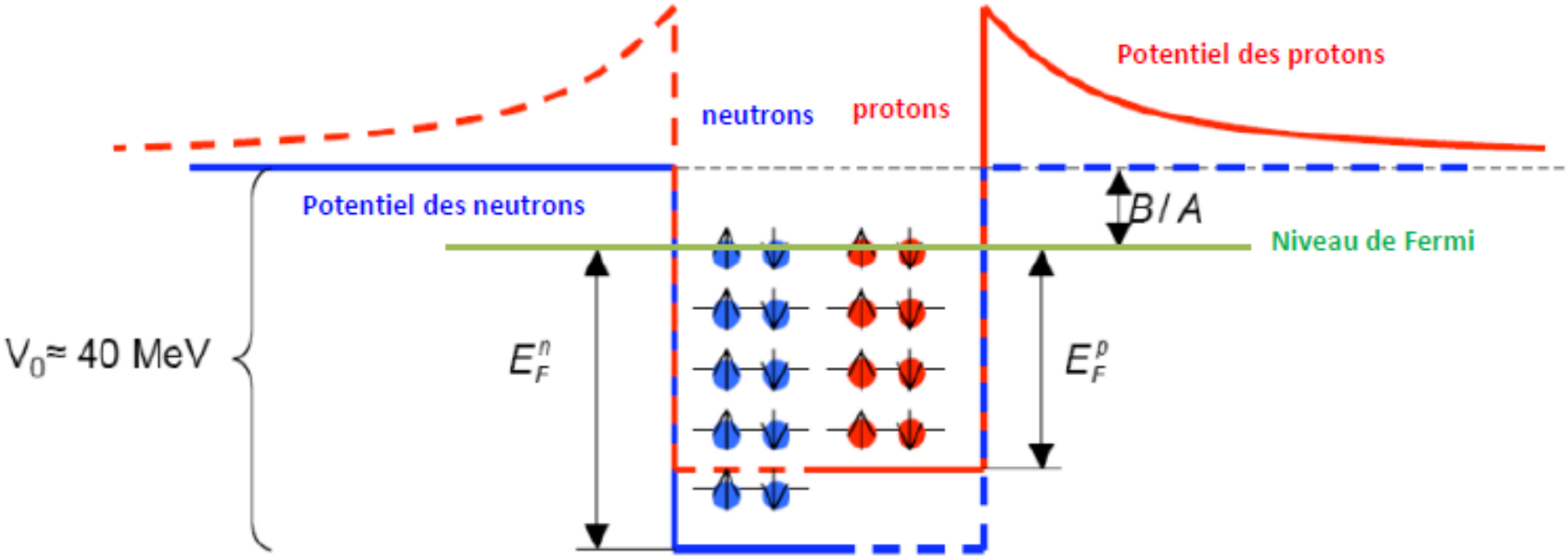
- E_F is independent on $A \rightarrow$ saturation property of nuclear force $\leftrightarrow$ each nucleon may only interact with a limited number of close nucleons
- The total depth of the well ≈ -(8 + 33) = -41 MeV is independent on A

Fermi-gas model: Potential well (6)

- Heavy nuclei have a surplus of neutrons → since Fermi level of protons and neutrons in a stable nucleus have to be equal (otherwise the nucleus would enter a more energetically favorable state through β decay) → the depth of the potential well as it is experienced by the neutron gas has to be larger than of the proton gas
- Protons are therefore on average less strongly bound in nuclei than neutrons → consequence of the Coulomb repulsion of the charged protons → extra term in the potential:

$$V_C = (Z-1)\frac{\alpha\hbar c}{R}$$

Fermi-gas model: Potential well (7)



Fermi-gas model: Asymmetry energy (1)

- The dependence of the binding energy on the surplus of neutrons (= Asymmetry energy) may be calculated within the Fermi gas model
- First we have to calculate ϵ the average kinetic energy per nucleon for $Z = N \rightarrow$

$$\epsilon = \frac{\int_0^{E_F} E dn}{\int_0^{E_F} dn} = \frac{\int_0^{E_F} E \frac{dn}{dE} dE}{\int_0^{E_F} \frac{dn}{dE} dE} = \frac{\int_0^{p_F} E \frac{dn}{dp} dp}{\int_0^{p_F} \frac{dn}{dp} dp}$$

- As $dn/dp = Const. \times p^2 \rightarrow$

$$\epsilon = \frac{\int_0^{E_F} E p^2 dp}{\int_0^{E_F} p^2 dp} = \frac{3p_F^2}{10m} = \frac{3}{5} E_F \simeq 20 \text{ MeV}$$

- The total Kinetic energy is thus $E_{kin}(N,Z) = A \times \epsilon$

Fermi-gas model: Asymmetry energy (2)

- for $Z \neq N \rightarrow E_{kin}(N,Z) = Z\epsilon_p + N\epsilon_n \rightarrow \epsilon = \frac{3}{10m}[N(p_F^n)^2 + Z(p_F^p)^2]$
- Considering →
- We obtain → $$Z = \frac{(p_F^p)^3 V}{3\pi^2\hbar^3} \text{ and } N = \frac{(p_F^n)^3 V}{3\pi^2\hbar^3}$$

$$\epsilon = \frac{3}{10m}\frac{\hbar^2}{R_0^2}\left(\frac{9\pi}{4}\right)^{2/3}\frac{N^{5/3}+Z^{5/3}}{A^{2/3}}$$

- In this expression → we assume again that the proton and neutron potential wells have the same radius
- This average kinetic energy has a minimum at *N* = *Z* for fixed mass number *A* (but varying *N* or *Z)* → Hence the binding energy gets maximal for *N* = *Z*

Fermi-gas model: Asymmetry energy (3)

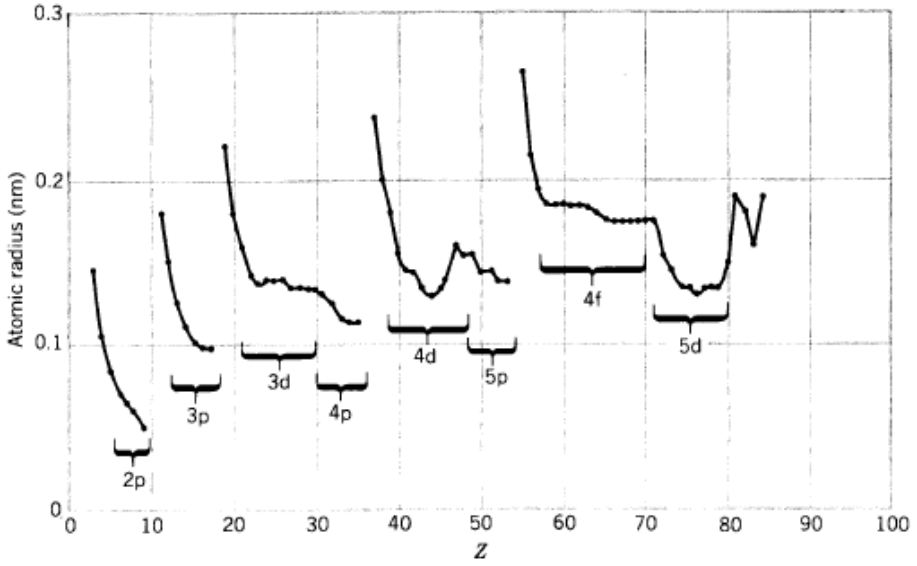
- Generally $N > Z$ → previous expression may be expanded in the difference N-Z →

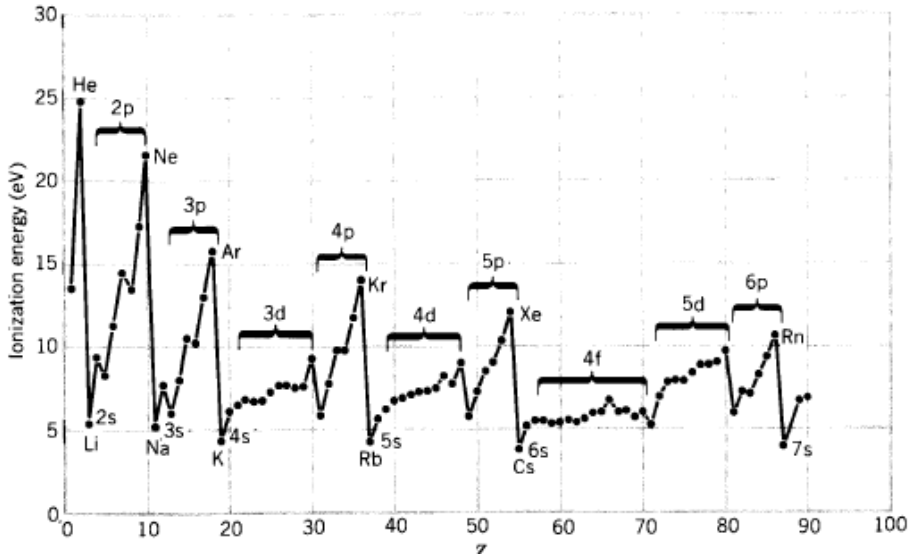
$$\epsilon = \frac{3}{10m}\frac{\hbar^2}{R_0^2}\left(\frac{9\pi}{8}\right)^{2/3}\left(A + \frac{5}{9}\frac{(N-Z)^2}{A} + \ldots\right)$$

- The first term corresponds to the volume energy in the Bethe-Weizsäcker mass formula
- The second term to the asymmetry energy
- The asymmetry energy grows with the neutron surplus → reducing of the binding energy

Shell model: Principles (1)

- Atomic theory based on the shell model provides remarkable clarification of the complicated details of atomic structure
- In the atomic shell model → regular and smooth variations of atomic properties *within* a subshell ↔ sudden and dramatic changes in the properties when changing of subshell
- Nuclear physicists attempted to use a similar theory for the problem of nuclear structure



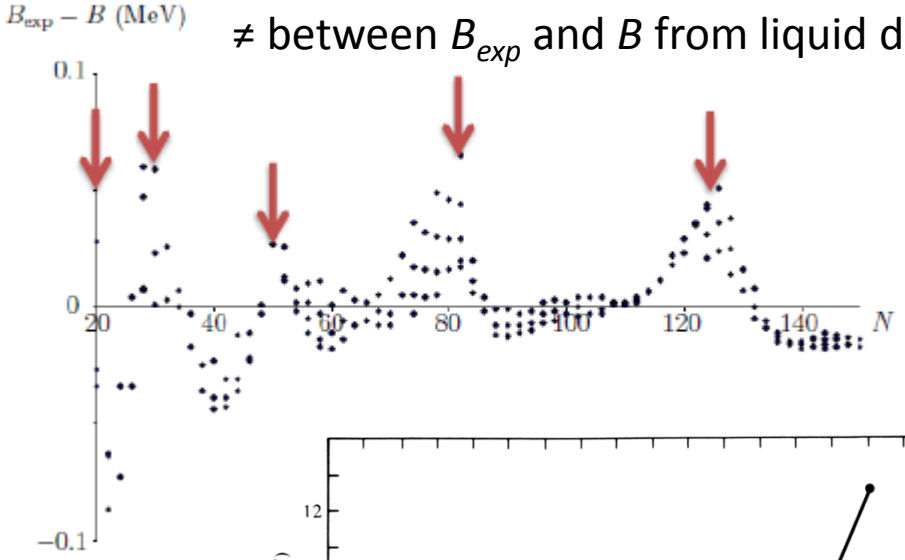


Shell model: Principles (2)

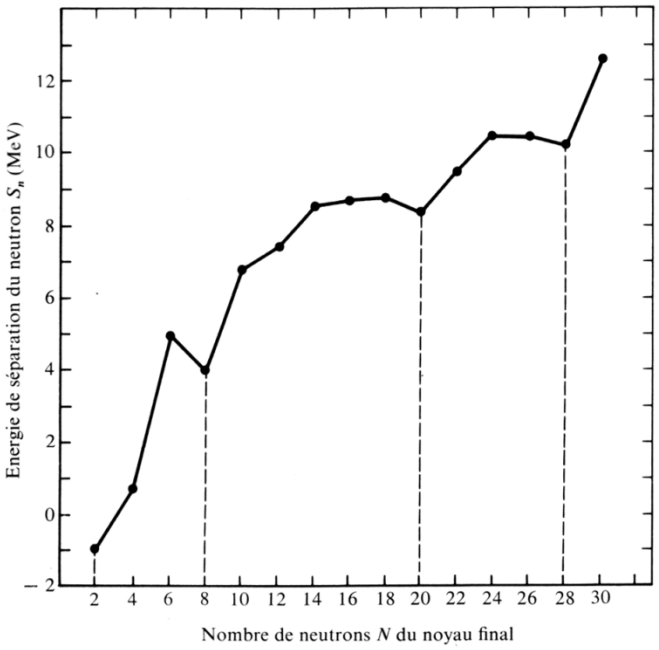
- In atomic case the potential is supplied by the Coulomb field of the nucleus ↔ in nuclear case the motion of a single nucleon is governed by a potential caused by all of the other nucleons → the nucleons move in a potential that they themselves create
- In atomic case atomic properties are described in terms of spatial orbits of the electrons → the electrons can move in those orbits without collision with other electrons ↔ in nuclear case collisions between nucleons would imply an energy transfer → but all energy levels are filled up to the level of the valence nucleons = high level → requires a lot of energy → impossible during nucleons collisions → collisions cannot occur → the nucleons can orbit as if they were transparent to one another

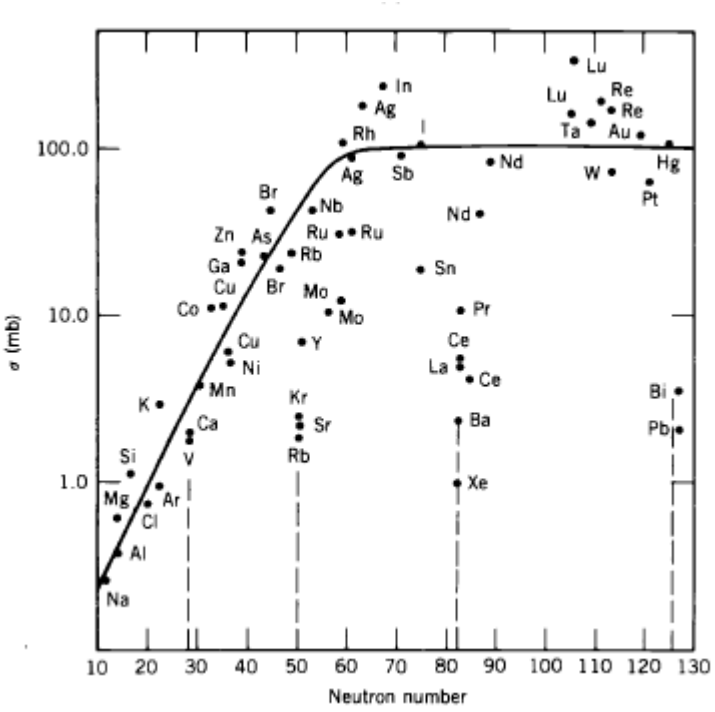
Shell model: Evidences (1)

≠ between B_{exp} and B from liquid drop model



Separation energy of the last neutron in a (*Z*,*N*+1) nucleus





Neutron-capture cross sections of various nuclei

Magic numbers are 2 – 8 – 20 – 28 – 50 – 82 – 126

Shell model: Simple potentials

- Solving the three-dimensional Schrödinger equation
- Simples choices of the potential →
  - Infinite well:
  - Harmonic oscillator:

$$V(r) = \begin{cases} 0 & \text{for} \quad r < R \\ \infty & \text{for} \quad r \geq R \end{cases}$$

$$V(r) = \frac{1}{2} m\omega^2 r^2$$

- In both cases → spherical symmetry → resolution of the radial part of the Schrödinger equation → achieving of energy states $E_{n\ell}$
- As in atomic physics → use of spectroscopic notation to label the levels ($n\ell$) → with one important exception → index *n* is *not* the principal quantum number → simply counts the number of levels with that ℓ value → 1d means the first (lowest) d state 2d means the second and so on

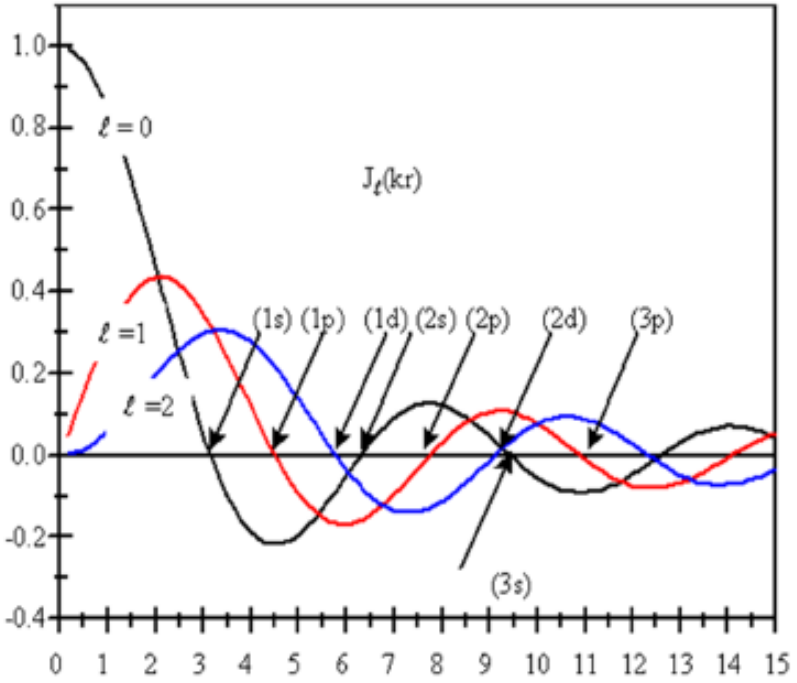
Shell model: Infinite well (1)

- For the Infinite well → the solution of the Schrödinger equation has the form $\Psi(r,\theta,\varphi) = \Psi_\ell(r)Y_{lm}(\theta,\varphi)$
- We can substitute $\Psi_\ell(r) = R_\ell(r)/r$
- The radial part of the Schrödinger equation becomes

$$-\frac{\hbar^2}{2m}\frac{d^2R(r)}{dr^2} + \frac{\hbar^2}{r^2}\frac{l(l+1)}{2m}R(r) = ER(r)$$

- The solution of the differential equation $y''(r) + \lambda(r)y(r) = 0$ are the Bessel functions j_ℓ (kr) with $k^2 = (2m/\hbar^2)E_{n\ell}$
- Boundary condition for the surface → at $r = R \rightarrow \Psi(R,\theta,\varphi) = 0$ → restrictions on k in Bessel functions → j_ℓ (kR) = 0 → main quantum number *n* corresponding to the nodes of the Bessel function $X_{n\ell}$ → $kR = X_{n\ell}$ → $E_{n\ell} = (\hbar^2/2m)\, X_{n\ell}^2/R^2$

Shell model: Infinite well (2)



	$X_{n,\ell}$		
ℓ	n =1 (1er zéro)	n =2 (2ème zéro)	n =3 (3ème zéro)
0	3.14 (1s)	6.28 (2s)	9.42 (3s)
1	4.49 (1p)	7.72 (2p)	10.90 (3p)
2	5.76 (1d)	9.09 (2d)	12.32 (3d)
3	6.98 (1f)	10.41 (2f)	

- Degeneracy of each level → the number of nucleons that can be put in each level = $2(2\ell + 1) \leftrightarrow (2\ell + 1)$ arises from the m_ℓ degeneracy (m_ℓ is integer value between $\pm\ \ell$) and the factor 2 comes from the m_s degeneracy (two possible spin directions)

Shell model: Infinite well (3)

state	$E_{nl} = C \cdot X_{nl}^2$	*degeneracy*	*states with* $E \leq E_{nl}$
$1s$	$E_{1s} = C \cdot 9.86$	2 $2\cdot(2l+1)$	2
$1p$	$E_{1p} = C \cdot 20.2$	6	8
$1d$	$E_{1d} = C \cdot 33.2$	10	18
$2s$	$E_{2s} = C \cdot 39.5$	2	20
$1f$	$E_{1f} = C \cdot 48.8$	14	34
$2p$	$E_{2p} = C \cdot 59.7$	6	40
$1g$	$E_{1g} = C \cdot 64$	18	58

- Only the first 3 magic numbers are reproduced
- Limitations of the model

Shell model: Harmonic oscillator (1)

- We consider a central oscillator potential $V(r) = (1/2)m\omega^2 r^2$
- Again we need only to consider the solution to the radial e
- As in the one-dimensional case → the solution is expressed as the product of an exponential and a finite polynomial

n	ℓ	E_n	$R(r)$
0	0	$\frac{3}{2}\hbar\omega_0$	$(2\alpha^{3/2}/\pi^{1/4})\, e^{-\alpha^2 r^2/2}$
1	1	$\frac{5}{2}\hbar\omega_0$	$(2\alpha^{3/2}\sqrt{2}/\sqrt{3}\,\pi^{1/4})(\alpha r)\, e^{-\alpha^2 r^2/2}$
2	0	$\frac{7}{2}\hbar\omega_0$	$(2\alpha^{3/2}\sqrt{2}/\sqrt{3}\,\pi^{1/4})(\frac{3}{2} - \alpha^2 r^2)\, e^{-\alpha^2 r^2/2}$
2	2	$\frac{7}{2}\hbar\omega_0$	$(4\alpha^{3/2}/\sqrt{15}\,\pi^{1/4})(\alpha^2 r^2)\, e^{-\alpha^2 r^2/2}$
3	1	$\frac{9}{2}\hbar\omega_0$	$(4\alpha^{3/2}/\sqrt{15}\,\pi^{1/4})(\frac{5}{2}\alpha r - \alpha^3 r^3)\, e^{-\alpha^2 r^2/2}$
3	3	$\frac{9}{2}\hbar\omega_0$	$(4\alpha^{3/2}\sqrt{2}/\sqrt{105}\,\pi^{1/4})(\alpha^3 r^3)\, e^{-\alpha^2 r^2/2}$
4	0	$\frac{11}{2}\hbar\omega_0$	$(4\alpha^{3/2}\sqrt{2}/\sqrt{15}\,\pi^{1/4})(\frac{15}{8} - \frac{5}{2}\alpha^2 r^2 + \frac{1}{2}\alpha^4 r^4)\, e^{-\alpha^2 r^2/2}$
4	2	$\frac{11}{2}\hbar\omega_0$	$(4\alpha^{3/2}\sqrt{2}/\sqrt{105}\,\pi^{1/4})(\frac{7}{2}\alpha^2 r^2 - \alpha^4 r^4)\, e^{-\alpha^2 r^2/2}$
4	4	$\frac{11}{2}\hbar\omega_0$	$(8\alpha^{3/2}/3\sqrt{105}\,\pi^{1/4})\,\alpha^4 r^4\, e^{-\alpha^2 r^2/2}$

Shell model: Harmonic oscillator (2)

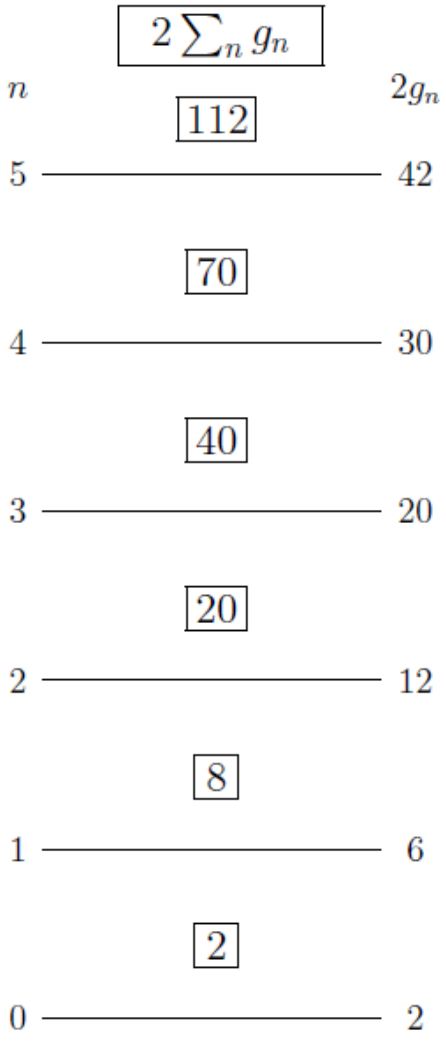
- The energy levels are given by →

$$E_n = \hbar\omega\left(n + \frac{3}{2}\right) \text{ where } n = n_x + n_y + n_z = 1, 2, 3, \ldots$$

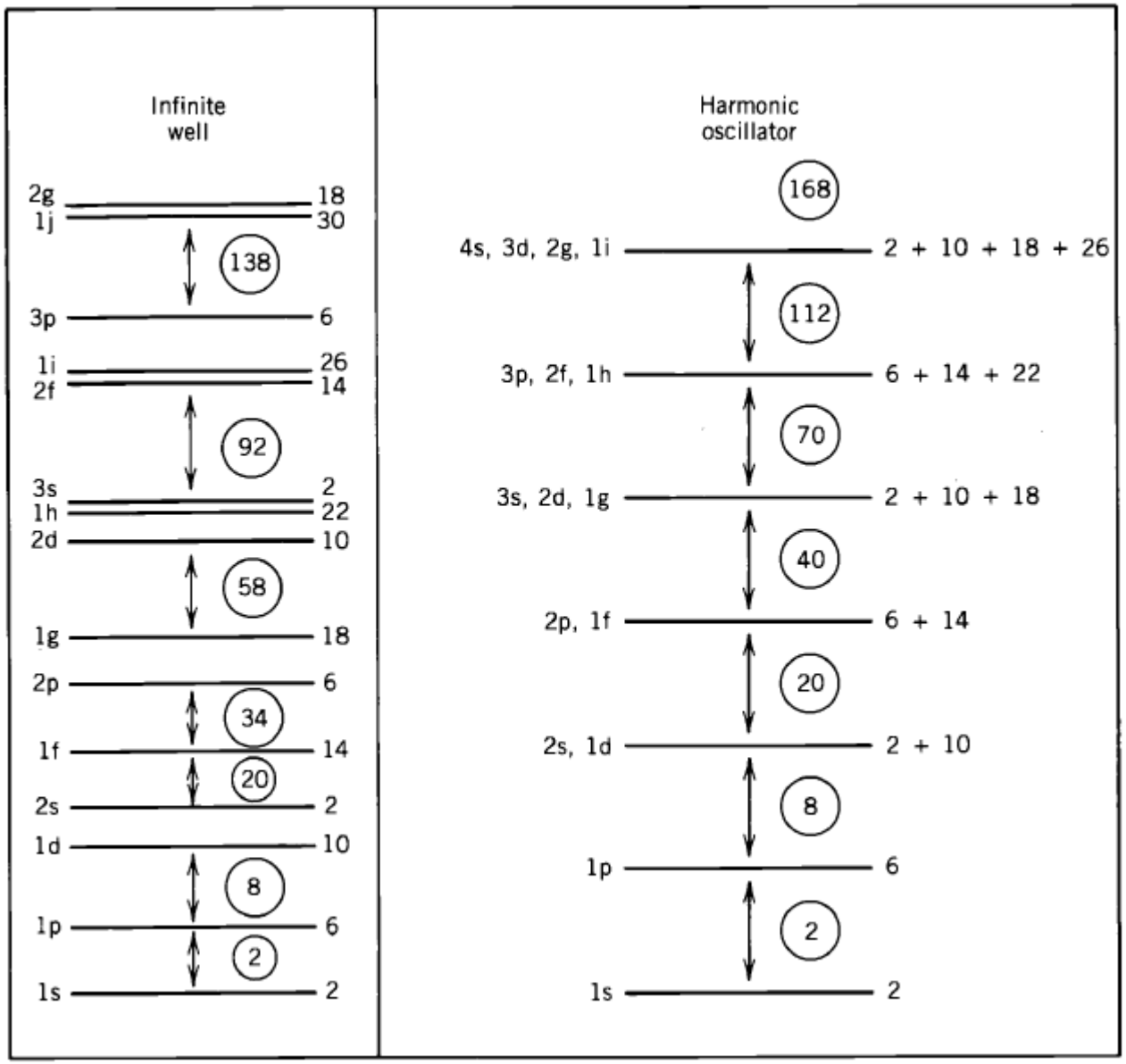
- The energy does not depend on ℓ → but not all ℓ values are allowed
- From the mathematical solution of the radial equation → restrictions: ℓ can be at most equal to *n* and takes only even or only odd values as *n* is even or odd
- Example: $n = 5 \rightarrow \ell = 1, 3, 5$
- Since the energies do not depend on m_ℓ → additional degeneracy of $2\ell + 1$ for each $\ell \leftrightarrow$ number of ways to obtain n with n_x , n_y and n_z
- Example: $n = 5 \rightarrow$ degeneracy of $[(2 \times 1 + 1) + (2 \times 3 + 1) + (2 \times 5 + 1)] = 21$
- Degeneracy → $\frac{1}{2}(n+1)(n+2)$

Shell model: Harmonic oscillator (3)

- m_s degeneracy factor 2 → total degeneracy: $(n + 1)(n + 2)$
- Again only the first 3 magic numbers are reproduced (2, 8, 20)



Shell model: Infinite well + Harmonic oscillator



- Shell structure obtained with infinite well and harmonic oscillator
- Capacity of each level is indicated to the right
- Large gaps between the levels → closed shells
- The circled numbers indicate the total number of nucleons at each shell closure
- No agreement with experiment

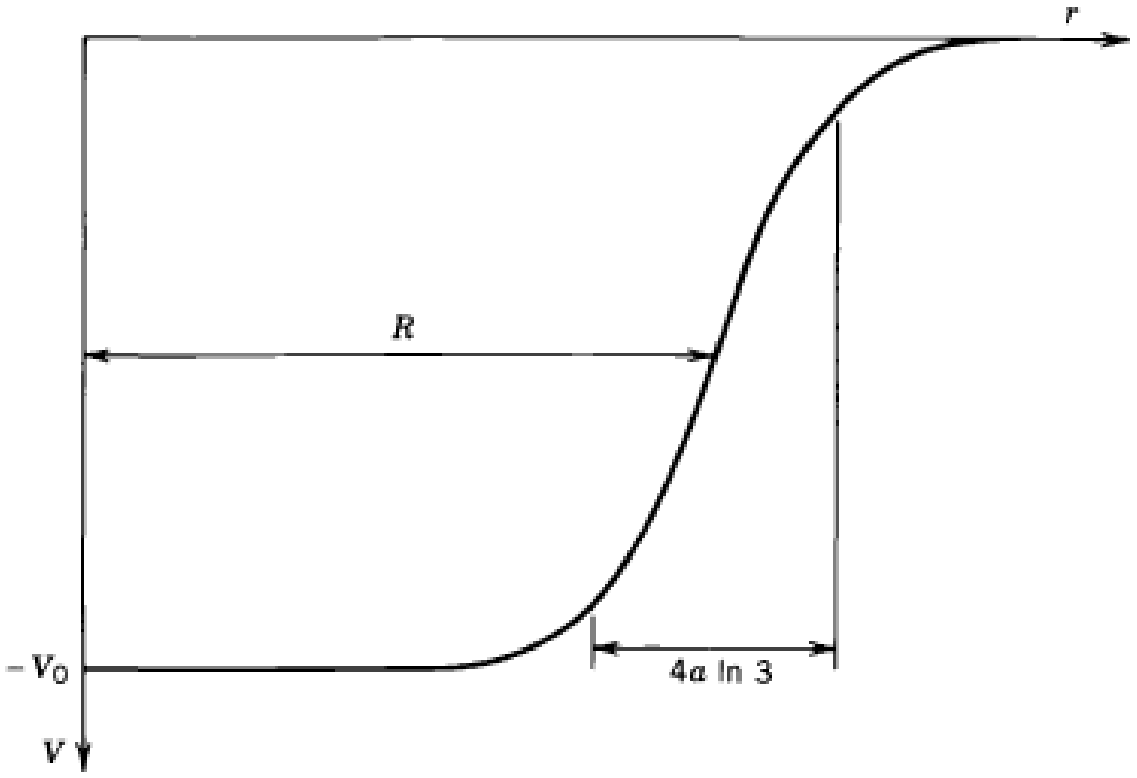
Shell model: Woods-Saxon potential (1)

- To improve the model → choice of a more realistic potential → infinite well and harmonic oscillator require infinite energy to separate neutron or proton
- Moreover infinite well shows too sharp edge while harmonic oscillator does not have a sharp enough edge
- Choice of ab intermediate form → the Woods-Saxon potential →

$$V(r) = \frac{-V_0}{1 + \exp\left[(r - R)/a\right]}$$

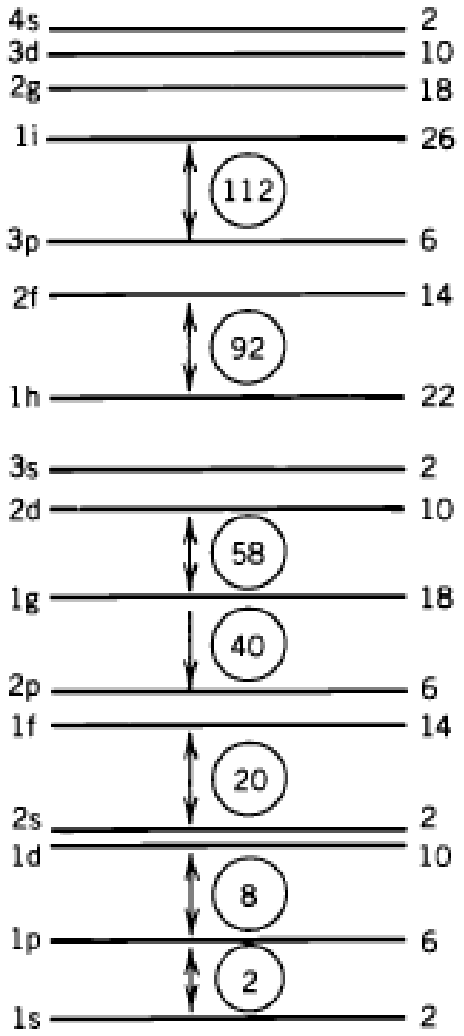
- $R = 1.24A^{1/3}$ is the mean radius and the $4a\ln 3 = 0.524$ fm is the skin thickness = the distance over which the potential changes from $0.9V_0$ to $0.1V_0$
- V_0 is adjusted to give the proper separation energies → ≈ 50 MeV

Shell model: Woods-Saxon potential (2)



Shell model: Woods-Saxon potential (3)

- Effect of the Woods-Saxon potential (compared with the harmonic oscillator) → remove the ℓ degeneracies of the major shells
- For increasing energy → the splitting becomes more and more severe → eventually as large as the spacing between the oscillator levels
- Filling the shells in order with $2(2\ell + 1)$ nucleons → we have again the magic numbers 2, 8 and 20 but not the higher



Shell model: Woods-Saxon + spin-orbit (1)

- To improve situation → no radical change in the potential → we do not want to destroy the physical content of the model → the nuclear potential should be like the Woods-Saxon potential → we have to add various terms to equation
- Mayer, Haxel, Suess and Jensen showed in 1949 that the inclusion of a spin-orbit contribution gives the proper separation of the subshells
- The spin-orbit interaction is written $\boldsymbol{V}_{so} = V_{ls}(r)\boldsymbol{\ell} \cdot \boldsymbol{s}$ → the form of $V_{ls}(r)$ is not important → the $\boldsymbol{\ell} \cdot \boldsymbol{s}$ factor causes the reordering of the levels
- The total angular momentum is defined by $\boldsymbol{j} = \boldsymbol{\ell} + \boldsymbol{s}$

Shell model: Woods-Saxon + spin-orbit (2)

- 1 single nucleon has s = ½ → the possible values of j are ℓ + ½ or ℓ - ½ (except for ℓ = 0 for which j = ½ is allowed)
- The expectation value $\langle \boldsymbol{\ell} \cdot \boldsymbol{s} \rangle$ is obtained from →

$$\boldsymbol{j}^2 = (\boldsymbol{l} + \boldsymbol{s})^2 = \boldsymbol{l}^2 + 2\boldsymbol{l} \cdot \boldsymbol{s} + \boldsymbol{s}^2 \Rightarrow \boldsymbol{l} \cdot \boldsymbol{s} = \frac{1}{2}(\boldsymbol{j}^2 - \boldsymbol{l}^2 - \boldsymbol{s}^2)$$

$$\langle \boldsymbol{l} \cdot \boldsymbol{s} \rangle = \frac{\hbar^2}{2}[j(j+1) - l(l+1) - s(s+1)]$$

$$V_{so} = \begin{cases} V_{ls}(r)\frac{\hbar^2}{2}l & \text{for } j = l + \frac{1}{2} \\ -V_{ls}(r)\frac{\hbar^2}{2}(l+1) & \text{for } j = l - \frac{1}{2} \end{cases}$$

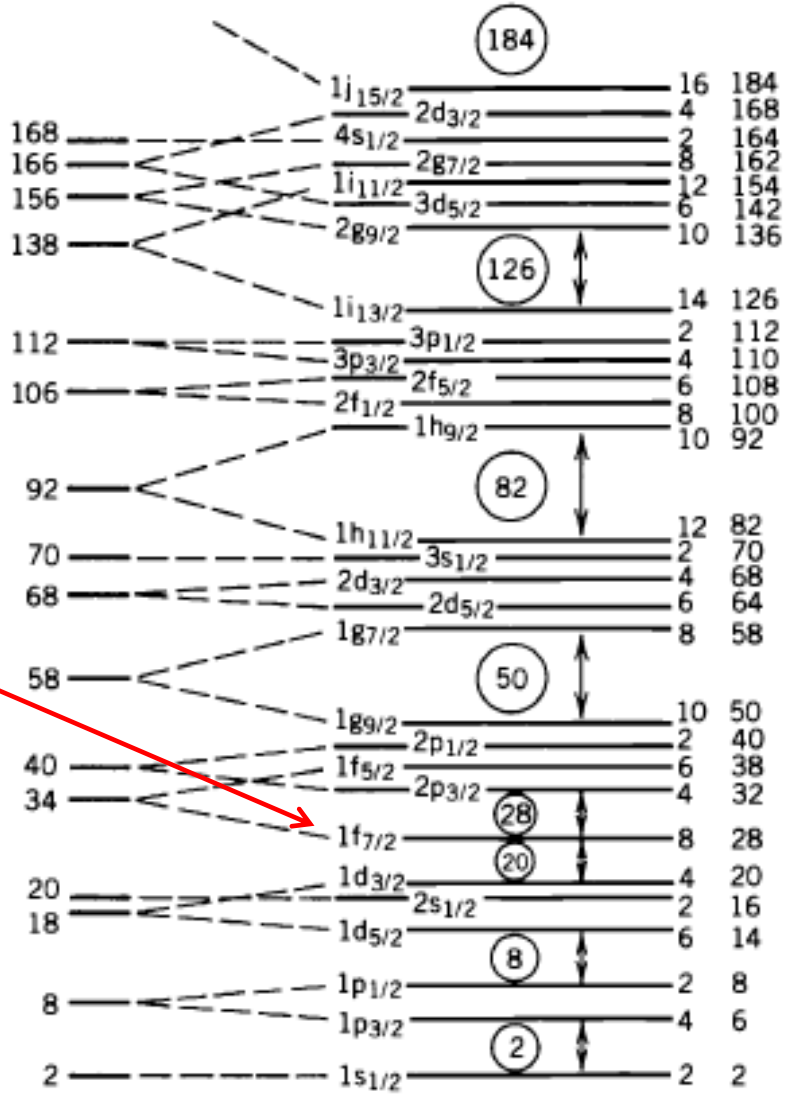
Shell model: Woods-Saxon + spin-orbit (3)

- We consider the 1*f* level ($\ell = 3$) with a $2(2\ell + 1)$ degeneracy = 14
- The possible *j* values are $\ell \pm ½ = 5/2$ or $7/2$ → presence of levels $1f_{5/2}$ and $1f_{7/2}$
- The degeneracy of each level is $(2j + 1)$ due to the m_j values ↔ with spin-orbit interactions m_l and m_s are no longer "good" quantum numbers → cannot be used to label states or to count degeneracies
- The capacity of the $1f_{5/2}$ level is 6 and 8 for $1f_{7/2}$ → total of 14 states as expected ↔ the number of possible states is preserved but they are grouped differently
- For the $1f_{7/2}$ and $1f_{5/2}$ the energy separation is $\propto$ the ≠ of $\langle \boldsymbol{\ell} \cdot \boldsymbol{s} \rangle$ for each state →

$$\langle \boldsymbol{l} \cdot \boldsymbol{s} \rangle_{j=l+1/2} - \langle \boldsymbol{l} \cdot \boldsymbol{s} \rangle_{j=l-1/2} = \frac{\hbar^2}{2}(2l+1)$$

Shell model: Woods-Saxon + spin-orbit (4)

- The energy splitting ↗ when ℓ ↗
- For $V_{ls}(r) < 0$ → the member of the pair with the larger j is pushed downward
- This level is now in the gap between the second and third shells → its capacity of 8 nucleons gives the magic number 28
- The p and d splittings do not result in any major regrouping of levels



Shell model: Woods-Saxon + spin-orbit (5)

- The next major effect of the spin-orbit term is on the 1*g* level → the $1g_{9/2}$ state is pushed down to the next lower major shell → its capacity of 10 nucleons is added to the previous total of 40 → magic number of 50
- Similar effect occurs at the top of each major shell → in each case the lower energy member of the spin-orbit pair from the next shell is pushed down into the lower shell → magic numbers appear as expected (even magic number not observed at 184)