# Chapter II:
# General properties of nuclei

# Summary

1. Notations, quantum numbers and spectra
2. Nuclear radius
3. Nuclear electromagnetic moments
4. Nuclear and atomic masses
5. Nuclei stability
6. Magic numbers
7. Particular types of nuclei

# Notations: Nuclide

- A nuclear species – *nuclide* – is defined by the number of neutrons *N* and by the number of protons *Z* (called the atomic number → charge in unit of *e*) → the mass number *A* is the total number of nucleons (the integer closest to the mass of the nucleus in *u*) → *A* = *N* + *Z* → with *X* the chemical symbol →

$$(A, Z) \leftrightarrow {}^{A}_{Z}X_{N} \leftrightarrow {}^{A}_{Z}X \leftrightarrow {}^{A}X$$

- Example: deuteron → $(2,1) \leftrightarrow {}^{2}_{1}H_{1} \leftrightarrow {}^{2}_{1}H \leftrightarrow {}^{2}H$
- *Isotopes* → have same charge *Z* but different *N*: ${}^{235}_{92}\mathrm{U}$ and ${}^{238}_{92}\mathrm{U}$
- *Isobars* → have the same mass number *A* → ${}^{3}_{2}\mathrm{He}$ and ${}^{3}_{1}\mathrm{H}$
- *isotone* → have the same *N* but different *Z* → ${}^{14}_{6}\mathrm{C}_{8}$ and ${}^{16}_{8}\mathrm{O}_{8}$

# Quantum numbers

- (A,Z) define a nuclear species → not the nuclear *quantum state*
- In atoms → individual electrons can move to higher energy orbits ↔ in nuclei → same for individual nucleons
- Nucleus (*A,Z*) has a rich spectrum of excited states (with few exceptions) which can decay to the ground state by emitting photons ($\gamma$-rays)
- Energy levels of a nucleus (including ground state) are characterized by *good quantum numbers* (integers or half-integers) corresponding to eigenvalues of operators (called *constants of motions*) commuting with the Hamiltonian *H* of the nucleus
- Constants of motions are deduced from symmetries of dominating interactions of nucleus (strong + Coulomb)

# Good quantum numbers

- Invariance under rotation → total angular momentum ***J*** = constant of motion
- Invariance under reflection (sometimes violated but weak violation in nuclei) → parity $\Pi$ = constant of motion
- Other good quantum number → total angular momentum projection $J_z$
- Complete set of commuting observables → {$H$, ***J***, $J_z$, $\Pi$}
- Nuclear levels are noted →

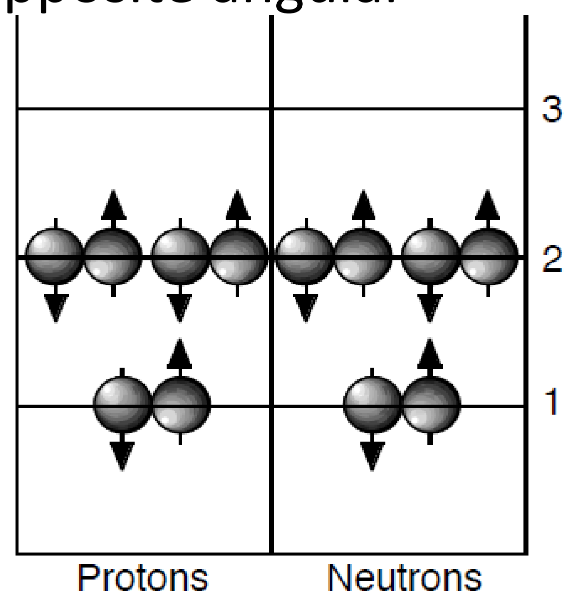
$$J^\pi \leftrightarrow J^\pi, E_x$$

- $J$ is the total angular momentum quantum number (spin), $\pi$ is the parity quantum number and $E_x$ is the excitation energy compared to the ground state
- Remark: $E_x$ does not depend on the quantum number $M$ associated to $J_z$ (different energy states for same $E_x$ and $J$)

# Quantum numbers: Ground state (1)

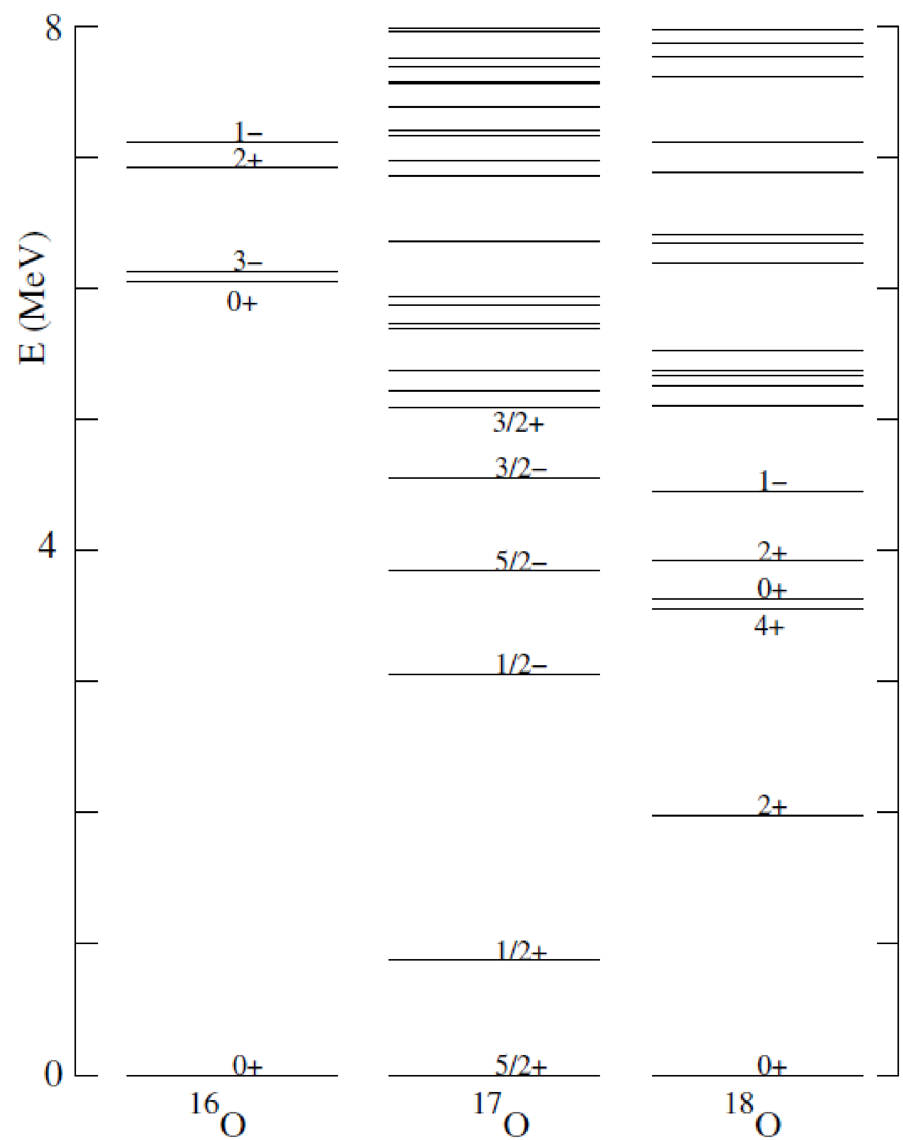
- Among all states of the nuclei → the most important is the ground state → some simple rules exist to determine its quantum state
- To obtain ground state → fill nucleons in lowest energy first
- To obtain ground state → pair up nucleons as you add them (« Katz's rule »)
- The ground state of all *N*-even and *Z*-even stable nuclei is characterized by the quantum numbers $0^+$ ↔ identical nucleons tends to pair with another nucleon of the opposite angular momentum → $J = 0$
- The parity is a statement about what the nuclear structure of the state would look like if the spatial coordinates of all the nucleons were reversed → $\pi = +$ means the reversed state = the original ↔ if even-even nucleus → $\pi = +$

$^{12}C$



# Quantum numbers: Ground state (2)

- The ground state of odd-*A* nuclei (even number of a kind of nucleon and odd number of the other kind) is described by the spin and parity of that single odd nucleon
- Remark: Prediction is correct if we recognize that single hole in subshell gives the same *J* and $\pi$ as single nucleon in same subshell



# Quantum numbers: Ground state (3)

- For odd-proton/odd-neutron nucleus → rules of Brennan and Bernstein (based on the shell model) →

- Rule 1: when $j_1 = l_1 \pm ½$ and $j_2 = l_2 \mp ½$ → $J = |j_1 - j_2|$
- Rule 2: when $j_1 = l_1 \pm ½$ and $j_2 = l_2 \pm ½$ → $J = |j_1 \pm j_2|$
- Rule 3: states that for configurations in which the odd nucleons are a combination of particles and holes → $J = j_1 + j_2 - 1$

- Parity is given by $\pi = -1^{(l_1 + l_2)}$

# Quantum numbers: Ground state (4)

Examples of application of the rules of Brennan and Bernstein:

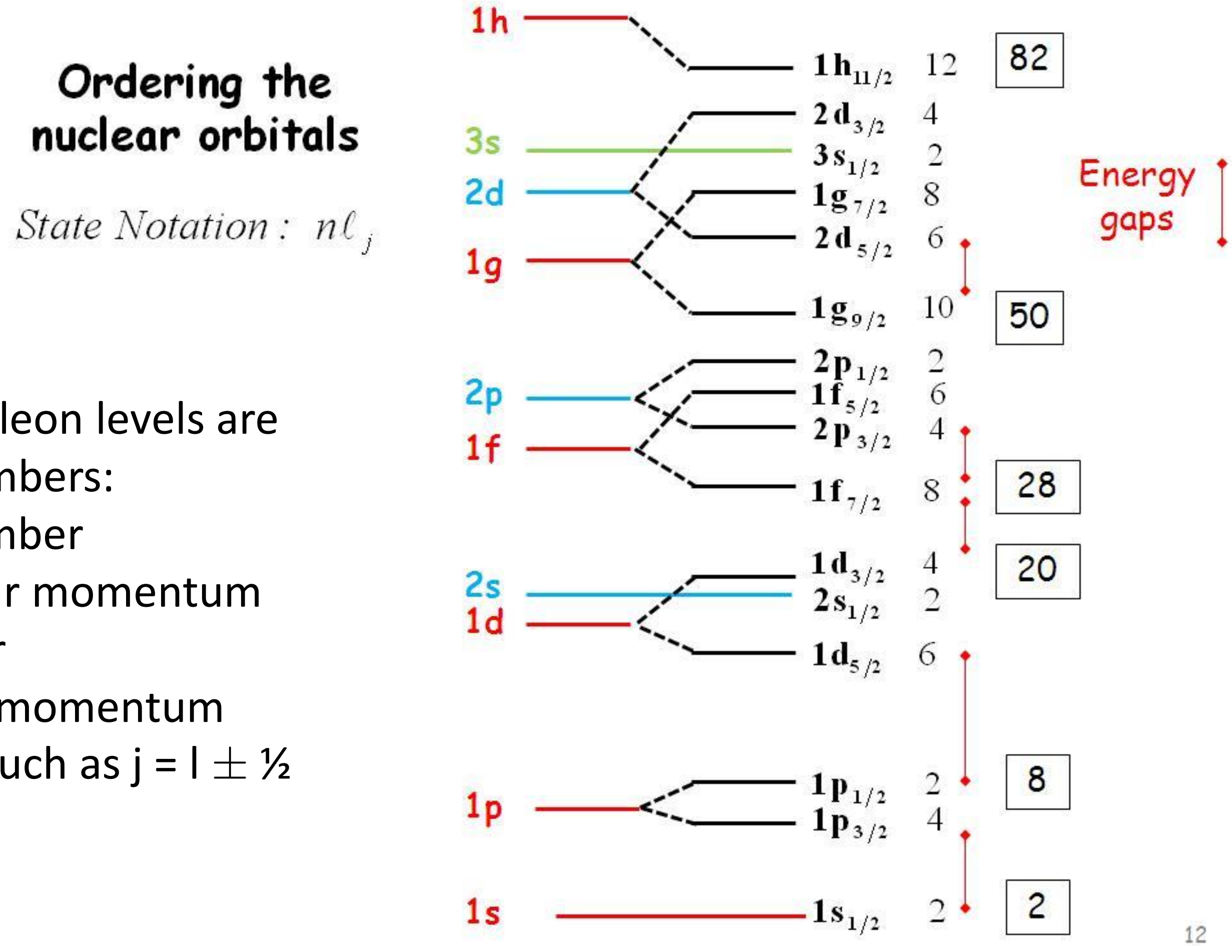
- $^{38}$Cl: 17 protons and 21 neutrons → the last proton is a $d_{3/2}$ level and the last neutron in a $f_{7/2}$ level →

  $j_p = 2 - ½$ / $j_n = 3 + ½$ → $J = |7/2 – 3/2| = 2$ / $\pi = -$
- $^{26}$Al: 13 protons and 13 neutrons → the last proton and neutron are in $d_{5/2}$ hole states →

  $j_p = j_n = 2 + ½$ → $J = |5/2 + 5/2| = 5$ / $\pi = +$
- $^{56}$Co: 27 protons and 29 neutrons → the last proton is in a $f_{7/2}$ hole state and the last neutron is in a $p_{3/2}$ state →

  $J = 7/2 + 3/2 - 1 = 4$ / $\pi = +$

# Energy level pattern for nucleons




For shell model → nucleon levels are characterized by 3 numbers:

- n: the principal number
- l: the orbital angular momentum quantum number
- j: the total angular momentum quantum number such as j = l ± ½

# Approximated good quantum numbers (1)

- Strong nuclear interaction → charge independence → particles affected equally by the strong force but with different charges (protons and neutrons) can be treated as different states of the same particle: the nucleon with a particular quantum number: the **isospin** (isotopic/isobaric spin) → value related to the number of charge states
- For a nucleon: 2 states → isospin quantum number $t = ½$ → 2 projections of the isospin → proton (p) has $m_t = -½$ and a neutron (n) has $m_t = +½$ (these projections are measured with respect to an arbitrary axis called the « 3-axis » in a system 1,2,3) → $t_3 = m_t\hbar$

# Approximated good quantum numbers (2)

- Interpretation of isospin → the operator

$$q = e(\frac{1}{2} - t_3)$$

gives the charge $e(1/2 - m_t)$ of the nucleon

- Definition of raising and lowering operators →

$$\begin{aligned} t_+ &= t_1 + it_2 \\ t_- &= t_1 - it_2 \end{aligned}$$

⇒

$$\begin{aligned} t_+ \left|p\right\rangle &= \left|n\right\rangle \\ t_+ \left|n\right\rangle &= 0 \\ t_- \left|n\right\rangle &= \left|p\right\rangle \\ t_- \left|p\right\rangle &= 0 \end{aligned}$$

# Approximated good quantum numbers (3)

- We define the total isospin *T* of a nucleons system as →

$$T = \sum_{j=1}^{A} t_j$$

- All properties of angular momentum can be applied to *T* →

$$\begin{aligned} T^2 \left| T M_T \right\rangle &= T(T+1) \left| T M_T \right\rangle \\ T_3 \left| T M_T \right\rangle &= M_T \left| T M_T \right\rangle \end{aligned}$$

- The total charge operator Q of the system can be written →

$$Q = e(\frac{1}{2}A - T_3) \quad \Longrightarrow \quad M_T = \frac{1}{2}A - Z = \frac{1}{2}(N - Z)$$

# Approximated good quantum numbers (4)

- For a system of nucleons → isospin follows same rules than ordinary angular momentum vector → 2-nucleons system has total isospin *T* = 0 or 1 (corresponding to antiparallel or parallel orientations of the 2 isospins) → the 3-axis component of the total isospin vector $T_3$ is the sum of the 3-axis components of the individual nucleons
- For any nucleus →

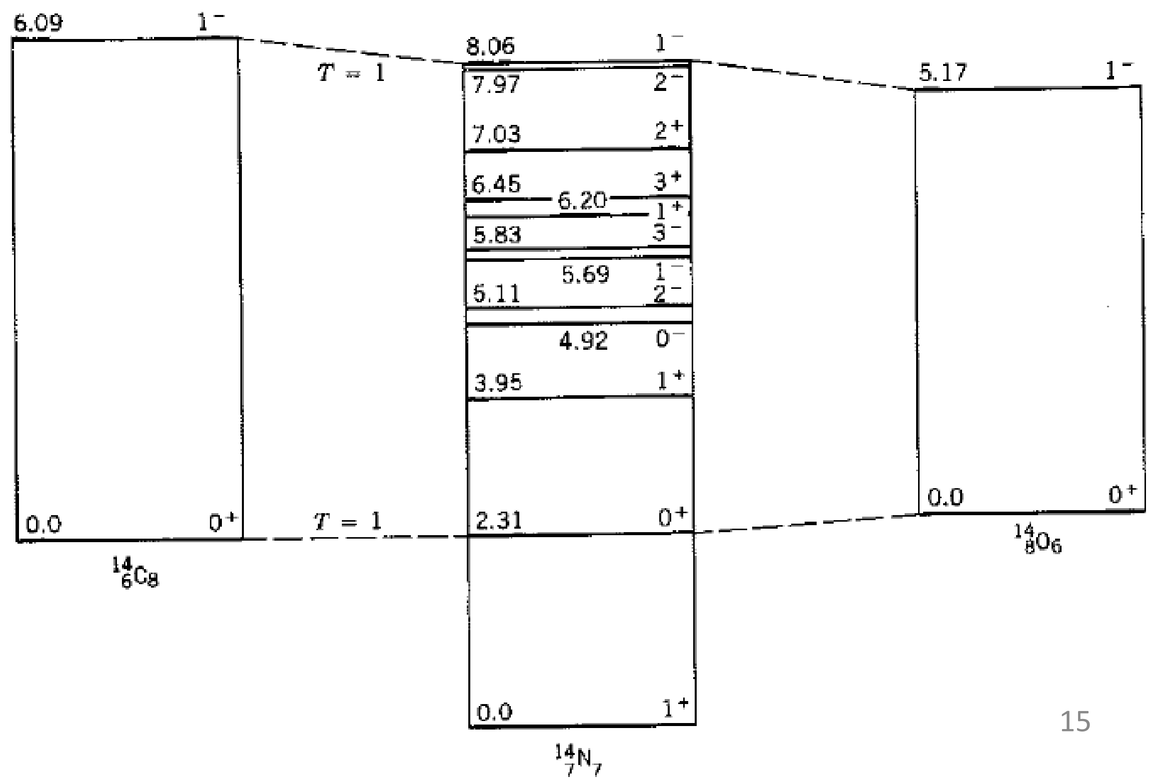
$$T_3 = \frac{1}{2}(N - Z) \leftrightarrow T \geq \frac{1}{2}|N - Z|$$

- Example: 2-nucleons system → p-p: $T_3 = -1$ ($T = 1$), n-n: $T_3 = +1$ ($T = 1$), p-n: $T_3 = 0$ ($T = 0$ or $T = 1$)

# Approximated good quantum numbers (5)

- If perfect charge independence (and electromagnetic interaction is not considered) → the isospin quantum number *T* gives the number *2T + 1* of isobars with this particular level in their spectrum with same quantum numbers *J* and $\pi$ → notation:

$$J^\pi; T \leftrightarrow J^\pi; T, E_x$$



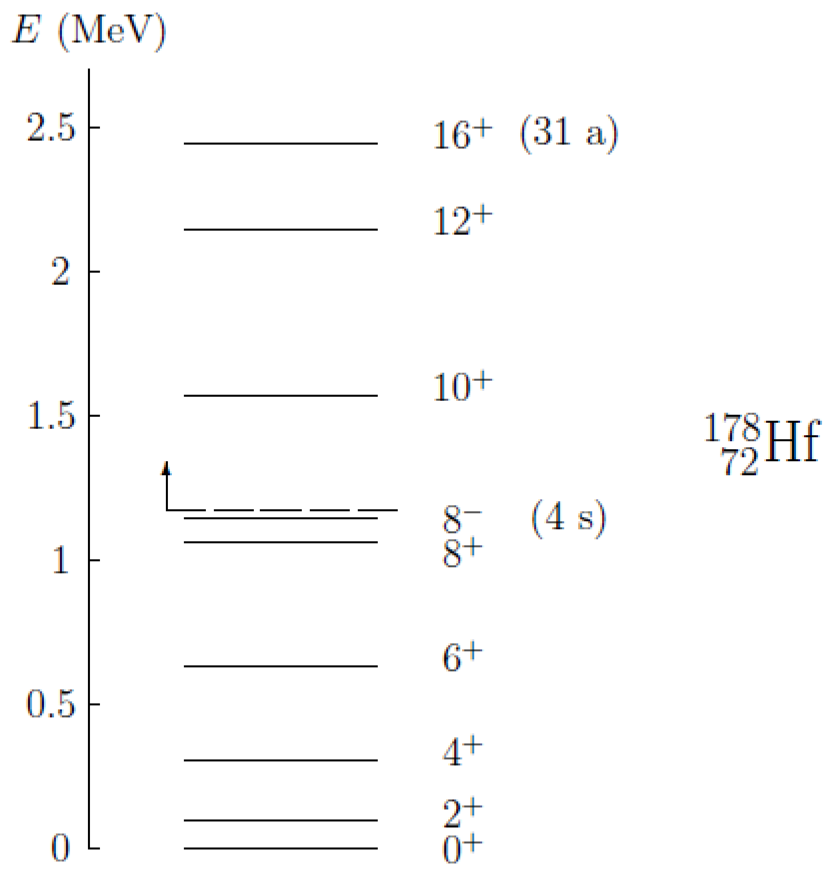

Energy levels of $^{14}C$ and $^{14}O$ are shifted by 2.36 and 2.44 MeV ↔ ≠ $m_{p,n}$ + Coulomb → $^{14}C$ and $^{14}O$ have *T* = 1 while $^{14}N$ has *T* = 0 except *T* = 1 for levels at 2.31 and 8.06 MeV

# Spectrum: Nuclear levels (1)

- ≠ types of energy levels
- Some are bound states → no spontaneous dissociation → de-excitation to levels with smaller energy by emitting radiation
- Some are resonances → there are beyond the dissociation threshold → dissociation or de-excitation
- Lifetimes of nuclear excited states are typically in the range $10^{-15} - 10^{-14}$ s → with few exceptions only nuclei in the ground state are present on Earth.
- The rare excited states with large lifetimes (> 1 s) are called isomeric states (or isomers or metastable states) → isomeric state of nucleus $^{A}X$ is designed by $^{Am}X$
- Isomer has generally $J$ and $\pi$ very different from states with smaller energy

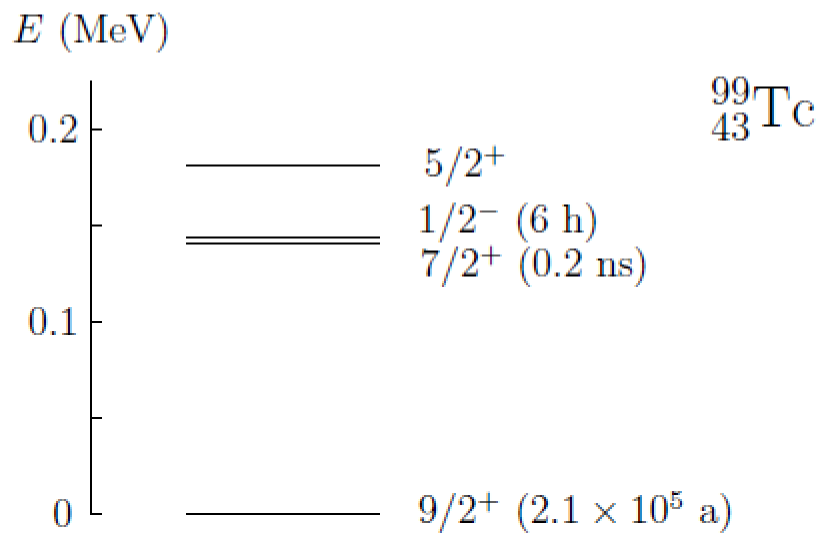
# Spectrum: Nuclear levels (2)

- Ground state $0^+$ = stable
- Level $8^-$ (1.147 MeV): $\tau$ = 4 s
- Level $16^+$ (2.446 MeV): $\tau$ = 31 years (de-excitation to $12^+$ → ≠ of $4\hbar$ →small probability)



# Spectrum: Nuclear levels (3)

- Ground state $9/2^+$ = unstable but long
- Level $1/2^- \rightarrow 7/2^+$
- Level $7/2^+$ (0.141 MeV): quick de-excitation by $\gamma$ emission to $9/2^+ \rightarrow$ application in nuclear medicine




- Extreme example → the first exited state of $^{180}$Ta has a lifetime $\tau$ = $10^{15}$ years while the ground state β-decays with $\tau$ = 8 hours → All $^{180}$Ta present on Earth is therefore in the excited state

# Nuclear radius (1)

- In chapter 1 → definition of the charge radius →

$$\langle r^2 \rangle_{ch}^{1/2} = \sqrt{\int r^2 \rho_{ch}(\boldsymbol{r}) d\boldsymbol{r}}$$

- The charge density of a nucleon is measured from the analysis of high energy electrons elastically scattered from it ↔ distance ≈ 0.1 fm → reduced de Broglie wave length $\lambda/2\pi = \hbar/p \approx 0.1$ fm → $E \approx pc \approx 2000$ MeV
- Initial electron wave function: $\exp(i\boldsymbol{k}_i\boldsymbol{r})$ (free particle of momentum $\boldsymbol{p}_i = \hbar\boldsymbol{k}_i$); scattered electron (also free particle with momentum $\boldsymbol{p}_f = \hbar\boldsymbol{k}_f$): $\exp(i\boldsymbol{k}_f\boldsymbol{r})$
- As elastic collision → $|\boldsymbol{p}_i| = |\boldsymbol{p}_f|$

# Nuclear radius (2)

- According to the Fermi Golden Rule → probability of transition is $\propto$ to the square of *F* (with *F(0)* = 1)→

$$F(\boldsymbol{k}_i, \boldsymbol{k}_f) = \int \psi_f^* V(r) \psi_i d\boldsymbol{r} \leftrightarrow F(\boldsymbol{q}) = \int e^{i\boldsymbol{q}\boldsymbol{r}} V(r) d\boldsymbol{r}$$

- $\boldsymbol{q} = \boldsymbol{k}_i - \boldsymbol{k}_f$ is the momentum change of scattered electron
- *V(r)* depends on the nuclear charge density $Ze\rho_{ch}(\boldsymbol{r}')$ →

$$V(r) = -\frac{Ze^2}{4\pi\epsilon_0} \int \frac{\rho_{ch}(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} d\boldsymbol{r}'$$

- With $\boldsymbol{qr} = qr\sin\theta$ and integrating on $\boldsymbol{r}$ → normalized $F(\boldsymbol{q})$ (called *form factor*) is:

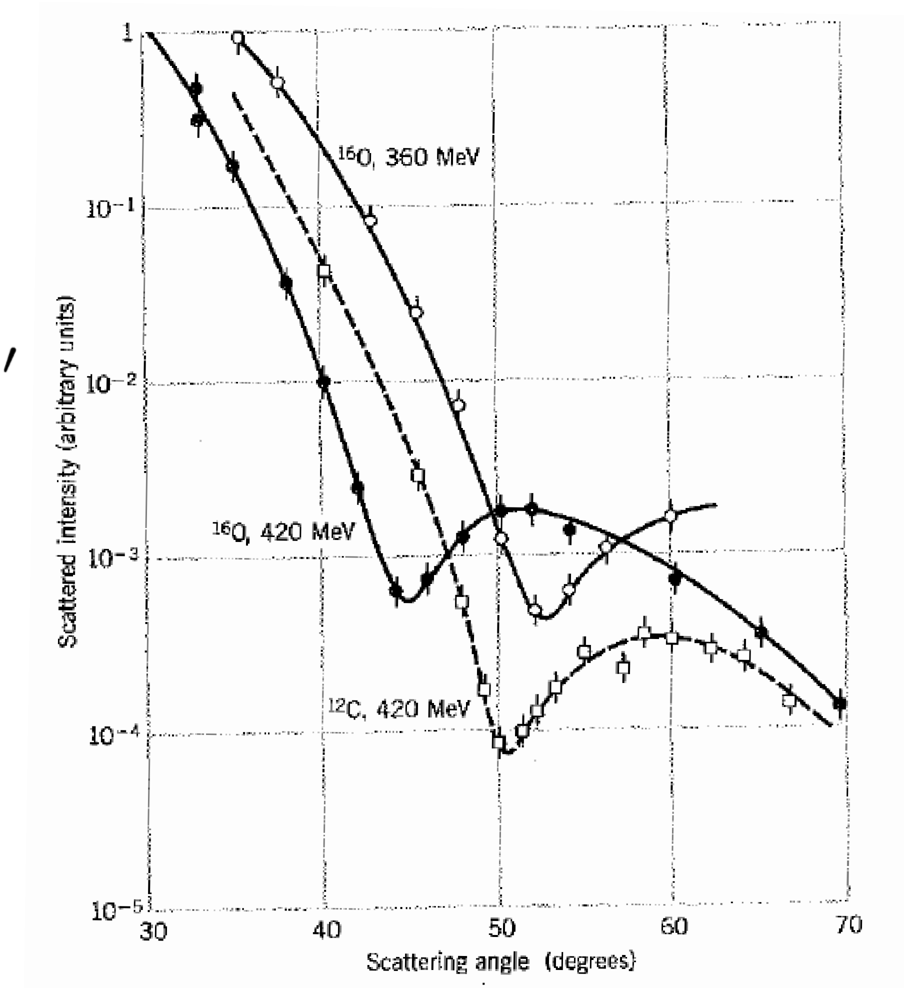
$$F(\boldsymbol{q}) = \int e^{i\boldsymbol{q}\boldsymbol{r}} \rho_{ch}(\boldsymbol{r}') d\boldsymbol{r}'$$

# Nuclear radius (3)

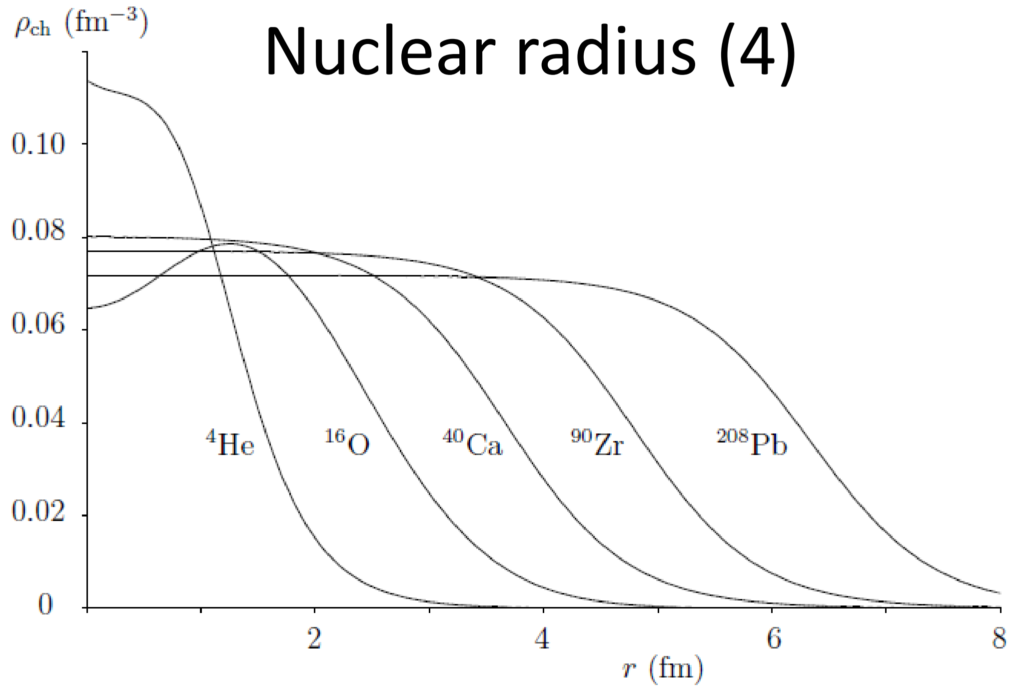
- If $\rho_{ch}(\boldsymbol{r'})$ depends only on $r'$ (not on $\theta'$ and $\phi'$) →

$$F(q) = \frac{4\pi}{q}\int \sin(qr')\rho_{ch}(r')r'dr'$$

- As | $\boldsymbol{p}_i$ | = | $\boldsymbol{p}_f$ | → q = f($\alpha$) with $\alpha$ the angle between $\boldsymbol{p}_i$ and $\boldsymbol{p}_f$ → $q = (2p/\hbar)\sin\alpha/2$ → the measure of $\alpha$ → $\rho_{ch}$



# Nuclear radius (4)




- Result for various nuclei → the central nuclear charge density is nearly the same for all nuclei → nucleons do not congregate at the center → nucleons are piled up as spheres ↔ short range of nuclear force
- The number of nucleons by unit volume is roughly constant → with *R* the mean nuclear radius of a sphere of uniform density with the same charge radius as the nucleus →

$$\frac{A}{4/3\pi R^3} \sim \text{ constant} \leftrightarrow R = R_0 A^{1/3}$$

# Nuclear radius (5)

- We can also define the matter radius *r* that is the root mean square radius of the distribution of nucleons such as →

$$\langle r^2 \rangle^{1/2} = \sqrt{\int r^2 \rho(\boldsymbol{r}) d\boldsymbol{r}}$$

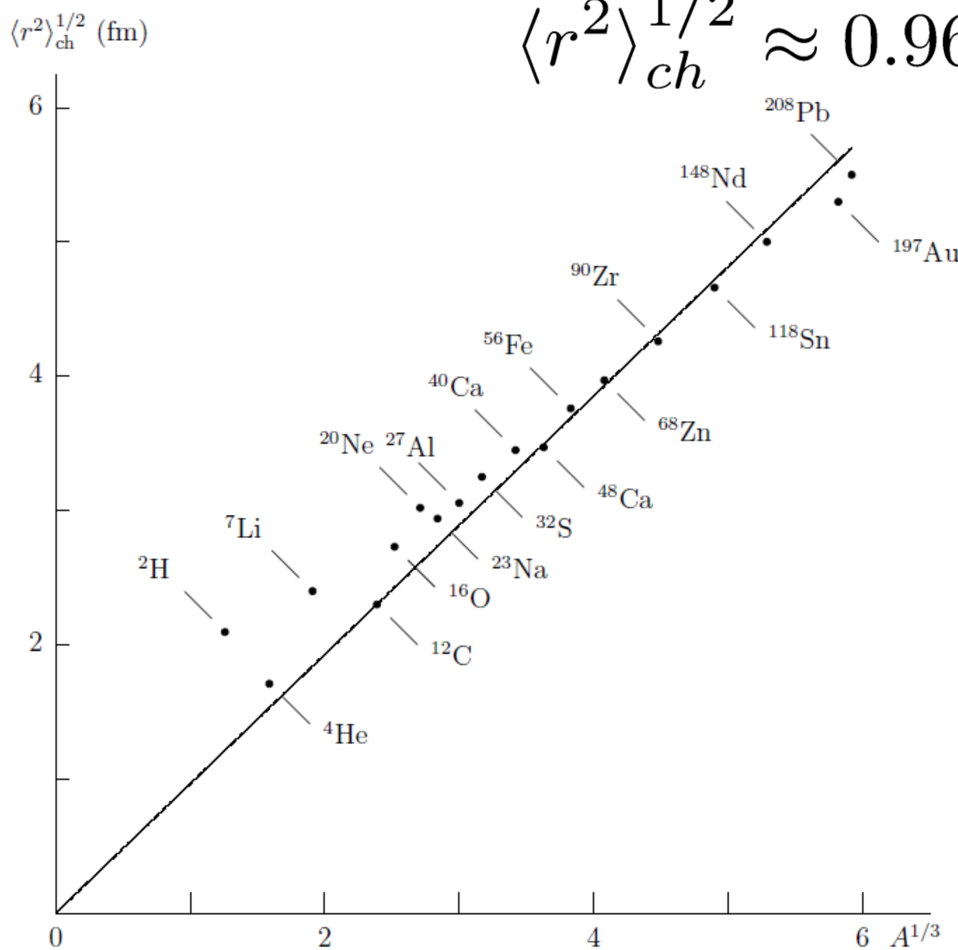
- For a sphere of constant density

$$\langle r^2 \rangle = \frac{\int r^2 \rho(\boldsymbol{r}) d\boldsymbol{r}}{\int \rho(\boldsymbol{r}) d\boldsymbol{r}} = \frac{\int_0^R r^4 dr}{\int_0^R r^2 dr} = \frac{3}{5} R^2$$

# Nuclear radius (6)

- From experimental measurements of $\rho_{ch}$ and considering that the charge radius also follows a law in $A^{1/3}$ →

$$\langle r^2 \rangle_{ch}^{1/2} \approx 0.96 A^{1/3}$$




Assuming that $r_{ch} \approx r$ (true for no-exotic nuclei) → $R_0$ = 1.24 fm

# Nuclear electromagnetic moments (1)

- Previous expressions are obtained for a sphere of constant density
- More precise calculations imply to consider a density obtained from the Wigner-Eckart theorem:

$$\rho(\boldsymbol{r}) = \sum_{\lambda \text{ pair}=0}^{2J} P_\lambda(\cos\theta_r)\rho^{(\lambda)}(r)$$

$$\rho^{(\lambda)}(r) = \frac{2\lambda+1}{4\pi r^2} \langle \Psi^{JJ\pi} | \sum_{\rho=1}^{Z} \delta(r'-r) P_\lambda(\cos\theta') | \Psi^{JJ\pi} \rangle$$

- The charge density is pair and has a rotational symmetry about *z*-axis
- For $J = 0 \rightarrow \rho(\boldsymbol{r}) = \rho^{(0)}(r)$

# Nuclear electromagnetic moments (2)

- For $J \neq 0 \rightarrow$ there is a measurable quantity which gives the difference between a spherical charge distribution and the real charge distribution → the electric-quadrupole moment
- In a general way the electric-multipole moment is written:

$$Q^{(\lambda)} = 2e \int r^{\lambda} P_{\lambda}(\cos\theta_r)\rho(\boldsymbol{r})d\boldsymbol{r} = \frac{8\pi e}{2\lambda + 1}\int_0^{\infty} r^{\lambda+2}\rho^{(\lambda)}(r)dr$$

- Electric-multipole moments are the moments of the charge density → as charge density is pair → only pair moments exist
- For $\lambda = 0 \rightarrow$ we obtain the trivial value $Q^{(0)} = 2Ze \propto$ to the total charge

# Nuclear electromagnetic moments (3)

- Due to the orthogonality of Legendre polynomials → multipole moment is ≠ 0 only if $J \geq \lambda/2$
- Moreover parity conservation implies → even $\lambda$
- Only even multipole moments lower or equal to *2J* give a non-zero value of $Q^{(\lambda)}$ →in particular electric-dipole moment of a nucleus is zero (considering that the parity is a good quantum number)
- The first non-trivial moment is the electric-quadrupole moment ($\lambda$ = 2)

# Electric-quadrupole moment

- The electric-quadrupole moment can be written:

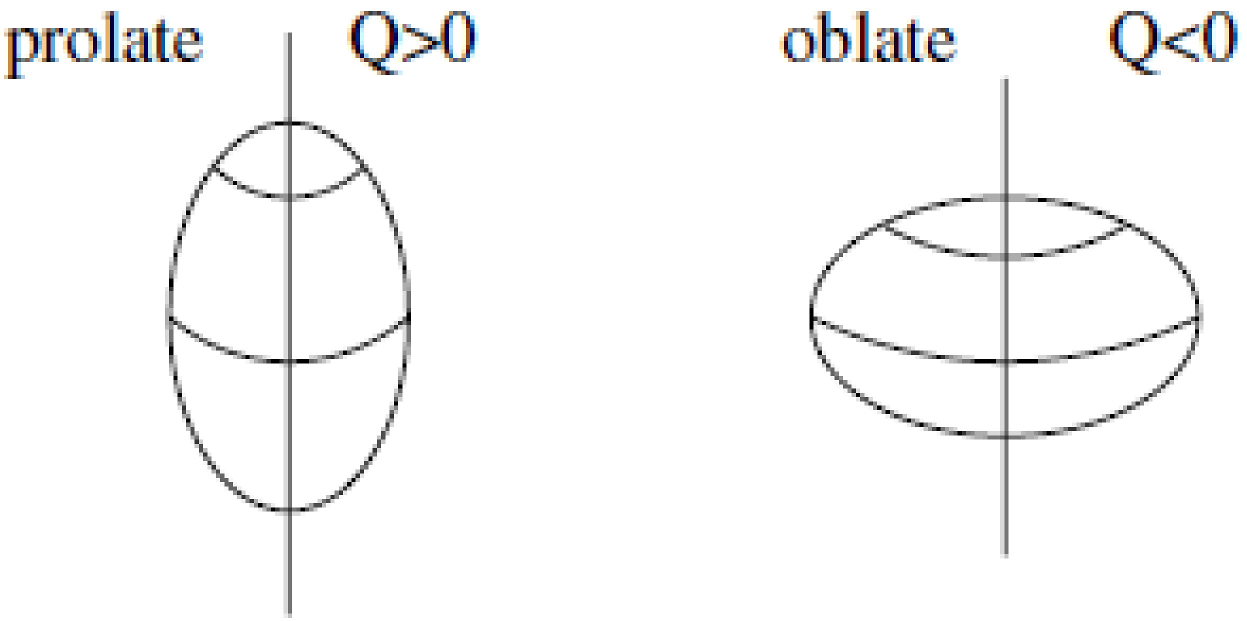
$$Q^{(2)} = 2e \int r^2 P_2(\cos\theta_r)\rho(\boldsymbol{r})d\boldsymbol{r}$$

- We can write

$$r^2 P_2(\cos\theta) = \frac{1}{2}(3z^2 - r^2) = \frac{1}{2}(2z^2 - x^2 - y^2)$$

- For $Q^{(2)} > 0$ → nucleus is deformed in the direction of $z$ → rugby ball shape (prolate)
- For $Q^{(2)} < 0$ → nucleus is deformed in the plane $\perp$ to $z$ → cushion shape (oblate)
- For $Q^{(2)} = 0$ → no deformation of the nucleus → spherical shape

# Nucleus deformation



# Magnetic-dipole moment (1)

- Magnetic-multipole moments are other moments characteristic of magnetic properties of the nucleus
- They come from the magnetization density
- The most important is the magnetic-dipole moment (simply called magnetic moment)
- The operator magnetic-dipole moment is

$$\boldsymbol{M} = \sum_{i=1}^{A} (g_{li} \boldsymbol{L}'_i + g_{si} \boldsymbol{S}_i)$$

with $g_{li}$ = 1 for a proton and 0 for a neutron, and $g_{si}$ are the gyromagnetic ratios ($g_{sp}$ = 5.5856947 and $g_{sn}$ = -3.826085)

# Magnetic-dipole moment (2)

- This operator is a combination of the operators $L'_i$ (orbital kinetic moment) and $S_i$ (spin) of each nucleon
- The magnetic-dipole moment is defined by (with $\mu_N$, the Bohr magneton):

$$\mu = \frac{\mu_N}{\hbar} \left\langle \Psi^{JJ\pi} \right| M_z \left| \Psi^{JJ\pi} \right\rangle$$

- From the Wigner-Eckart theorem → the nucleus has a magnetic moment if $J \geq 1/2$

# Atomic mass

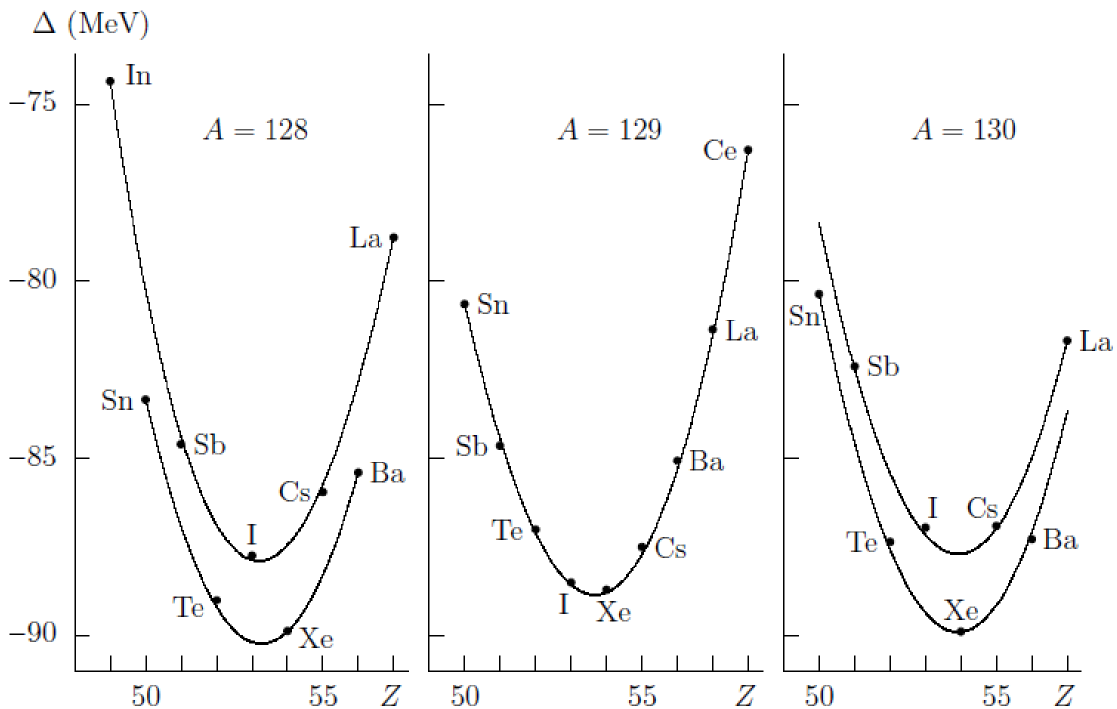
- Mass is bound to energy conservation → important to define stability of nuclei
- We define the *atomic mass*: mass of a neutral atom in ground state → $M(A,Z)$ or $M(^{A}X)$
- 1 unified *atomic* mass unit (u) = $1.6605390 \times 10^{-27}$ kg = 931.4940 MeV/$c^2$ = 1/12 of M(12,6) (*atom* of $^{12}$C)
- Example:
  - $M(^{1}H)$ = 1.007825032 u
  - $m_p$ = 1.007276467 u
  - $m_e$ = 5.48579909 $10^{-4}$ u
  - $M(^{1}H) = m_p + m_e$ - 1 Rydberg (electron binding energy must be considered)

# Mass excess

- Mass atomic is often given in *mass excess* form: *Δ(A,Z)* (energy expressed in MeV) →

$$\Delta(A,Z) = [M(A,Z) - A]uc^2$$

- *Δ* for isobar families (fixed A) as a function of Z varies only a little →




Parabolic in shape for odd-*A* and double parabola for even-*A*

# Nuclear mass

- The *nuclear mass* is the mass of the nucleus of an isotope in ground state → $m(A,Z)$ or $m(^{A}X)$
- We have →

$$m(A,Z) = M(A,Z) - Zm_e + B_e(Z)/c^2$$

- The electron binding energy $B_e$ decreases the total mass of the atom
- $B_e = \sum_i B_i$ with $B_i = a_i(Z-c_i)^2$ ($a_i$ and $c_i$ are constant parameters for each electron shell
- $B_e$ is generally neglected in the definition of nuclear mass (in first approximation) because it is quite smaller than usual nuclear energies ($B_e \sim$ 10-100 keV ↔ $M(A,Z) \sim A \times$ 1000 MeV)

# Binding energy of a nucleus (1)

- The binding energy *B* of a nucleus is defined as the negative of the difference between the nuclear mass and the sum of the masses of the constituents →

$$B(A,Z) = [Nm_n + Zm_p - m(A,Z)]c^2$$

- *B* is positive for all nuclei (stable or unstable) → implies that the nucleus does not spontaneously break down into its all constituents (but does not imply that it is stable)
- We can write ($^1$H is the hydrogen atom and $B_e(1)$ = 13.6 eV) →

$$\begin{aligned} B(A,Z) &= [Nm_n + ZM(^1H) - m(A,Z)]c^2 - B_e(Z)Z + B_e(1) \\ &\simeq [Nm_n + ZM(^1H) - m(A,Z)]c^2 \end{aligned}$$

- By adding and subtracting $A = N + Z$ →

$$B(A,Z) = N\Delta(^1n) + Z\Delta(^1H) - \Delta(A,Z)$$

# Separation energy

- Analogous to ionization energy in atomic physics → definition of the neutron/proton separation energy = amount of energy that it is needed to remove a neutron/proton from a nucleus ${}^{A}_{Z}X_N$

$$
\begin{aligned}
S_n &= B({}^{A}_{Z}X_N) - B({}^{A-1}_{Z}X_{N-1}) \\
&\simeq [m({}^{A-1}_{Z}X_{N-1}) - m({}^{A}_{Z}X_N) + m_n]c^2
\end{aligned}
$$

$$
\begin{aligned}
S_p &= B({}^{A}_{Z}X_N) - B({}^{A-1}_{Z-1}X_N) \\
&\simeq [m({}^{A-1}_{Z-1}X_N) - m({}^{A}_{Z}X_N) + m({}^{1}H)]c^2
\end{aligned}
$$

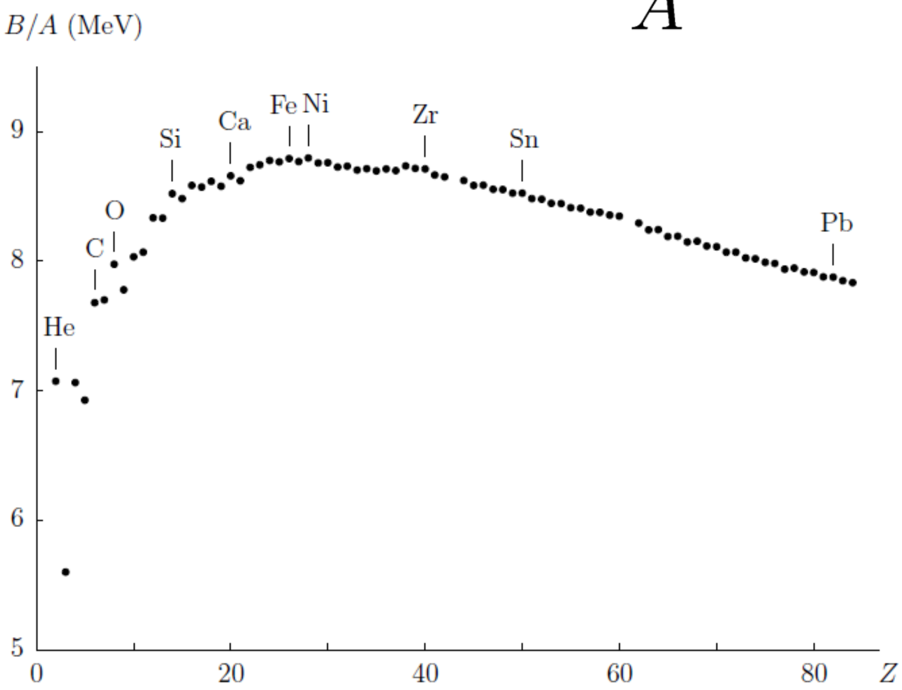
**Table 3.1** Some Mass Defects and Separation Energies

| Nuclide | Δ (MeV) | $S_n$ (MeV) | $S_p$ (MeV) |
|---|---|---|---|
| $^{16}$O | −4.737 | 15.66 | 12.13 |
| $^{17}$O | −0.810 | 4.14 | 13.78 |
| $^{17}$F | +1.952 | 16.81 | 0.60 |
| $^{40}$Ca | −34.847 | 15.64 | 8.33 |
| $^{41}$Ca | −35.138 | 8.36 | 8.89 |
| $^{41}$Sc | −28.644 | 16.19 | 1.09 |
| $^{208}$Pb | −21.759 | 7.37 | 8.01 |
| $^{209}$Pb | −17.624 | 3.94 | 8.15 |
| $^{209}$Bi | −18.268 | 7.46 | 3.80 |

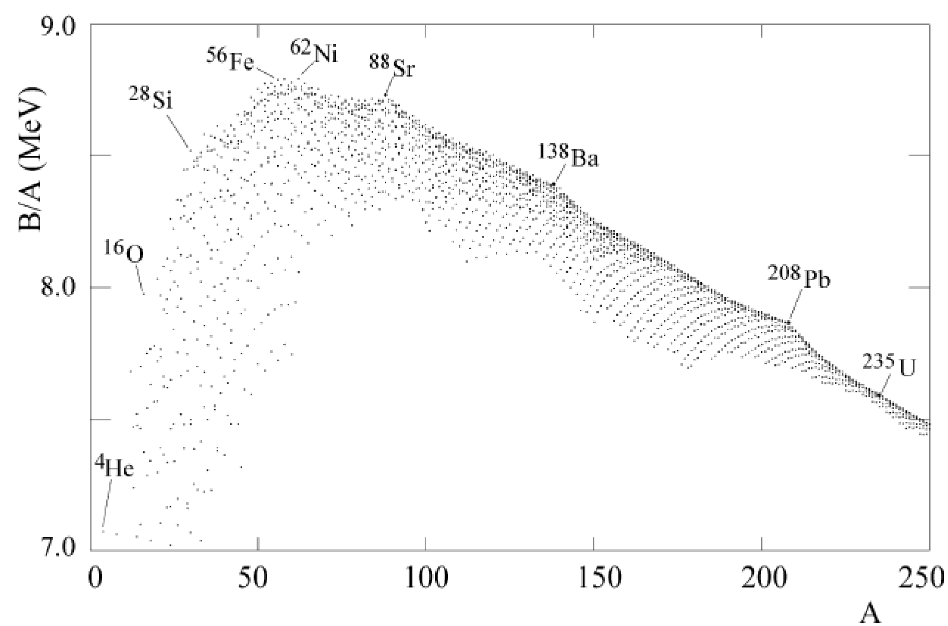
# Binding energy of a nucleus (2)

- As first approximation for stable nuclei (with $A \gtrsim 12$) →

$$\frac{B(A,Z)}{A} \approx (8.3 \pm 0.5)\ \mathrm{MeV}$$




Stable nuclei




Stable and unstable nuclei

# Binding energy of a nucleus (3)

- This property is explained by the short range of nuclear force
- Indeed for a long range force (as Coulomb force) → the binding energy of a *n*-particles system is $\propto$ to the number of particles pairs → $B(n) = (1/2)n(n-1)$
- As the binding energy of a nucleus as not this trend → nuclear interaction has the *saturation* property → each nucleon may only interact with a limited number of close nucleons
- The binding energy by nucleon is fixed by the numbers of neighbours → independent on the size of the nucleus
- If the nuclei is too small → saturation is not reached

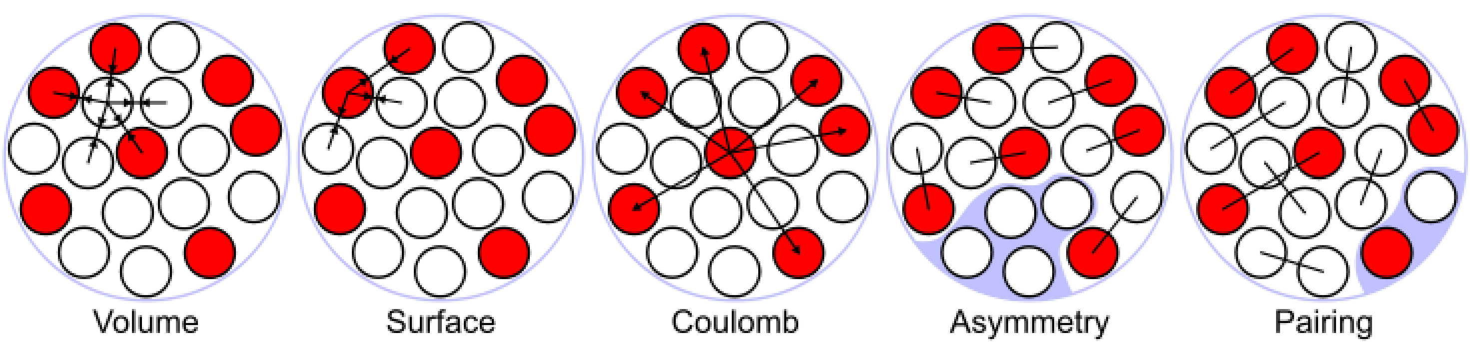
# Binding energy of a nucleus: Bethe-Weizsäcker formula

- Semi-empirical expression based on simplified physical arguments and on a fitting to data → valid for absolutely stable nuclei
- Physical model beyond it → liquid drop model → the nucleus is treated as a drop of incompressible nuclear fluid of very high density

$$B(A,Z) = a_V A - a_S A^{2/3} - a_C \frac{Z(Z-1)}{A^{1/3}} - a_a \frac{(N-Z)^2}{A} + \delta$$

$a_V \simeq 15.56$ MeV $\quad a_C \simeq 0.72$ MeV

$a_S \simeq 17.23$ MeV $\quad a_a \simeq 23.285$ MeV



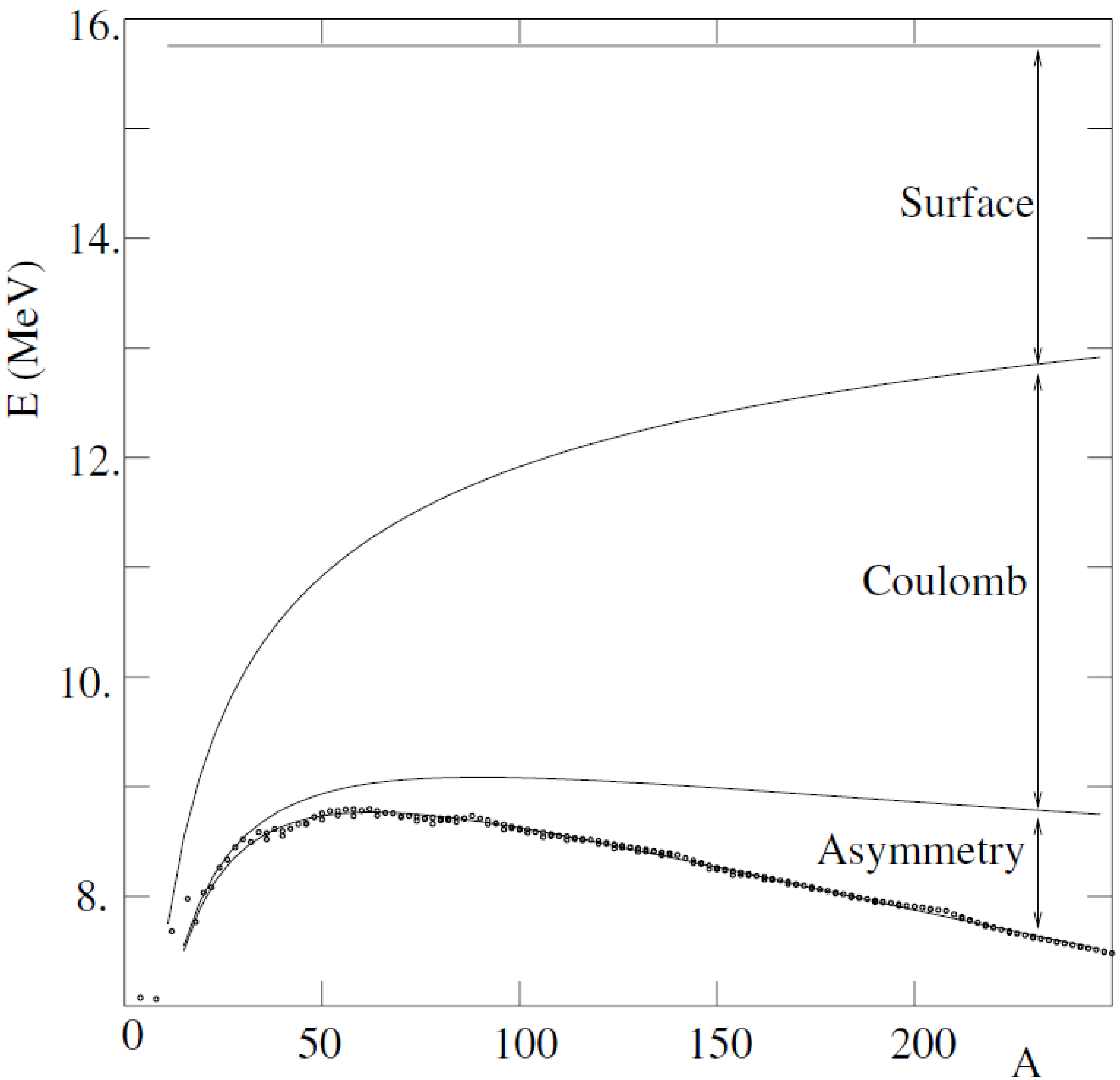
# Bethe-Weizsäcker formula (1)

- $a_V A$: Volume term → reflects the saturation property → each nucleon interacts only with nearest-neighbours → constant binding energy per nucleon $B/A$
- $-a_S A^{2/3}$: Surface term → lowers the binding energy → nucleons near the surface feel forces coming only from the inside of the nucleus → their contribution in first term is overestimated → $\propto$ to the area $4\pi R^2 \sim A^{2/3}$
- $-a_C Z(Z-1)A^{-1/3}$: Coulomb repulsion term → long range force due to protons → Coulomb energy $E_C$ of a sphere of charge $Ze$ and radius $R$ → $E_C = \frac{3}{5}\frac{Z^2 e^2}{4\pi\epsilon_0 R}$ → as this energy is $\propto$ to number of protons pairs → $Z^2$ must be replaced by $Z(Z-1)$ → $a_C = 0.6e^2/4\pi\epsilon_0 R \approx 0.72$ MeV (with $R = 1.24$ fm) → it favors a neutron excess over protons
- $-a_N(N-Z)^2 A^{-1}$: Asymmetry term → due to Pauli principle (isospin) the minimum energy in a nucleus is reached for $N \approx Z$ (otherwise we could have $Z = 2$ and $N = 100$ ) → if proton was not charged we would exactly $N = Z$ but due Coulomb repulsion $N \geq Z$ → for small $A$ → $N = Z$ and for large $A$ → $N > Z$ → asymmetry term $\propto$ to the difference between $N$ and Z → Fermi model gives $(N-Z)^2/A$

# Bethe-Weizsäcker formula (2)

- $\delta$: Pairing term → as seen before → nucleons have tendency to couple pairwise to establish stable configuration →
  - Odd $A$: $\delta = 0$ by definition → less favorable than N-even and Z-even but more favorable than $N$-odd and $Z$-odd
  - $Z$-even/$N$-even: all nucleons may be paired → the bonding is favored ($\delta > 0$) → empirical expression $\delta = +12A^{-1/2}$ MeV
  - $Z$-odd/$N$-odd: one neutron and one proton cannot be paired → the binding energy is decreased ($\delta < 0$) → empirical expression $\delta = -12A^{-1/2}$ MeV
- This pairing term has important consequences on the stability of nuclei → Among the 275 stable known nuclei → 166 are $Z$-even/$N$-even – 55 are $Z$-even/$N$-odd – 50 are $Z$-odd/$N$-even – 4 are $Z$-odd/$N$-odd: $^{2}H$, $^{6}Li$, $^{10}B$, $^{14}N$ (attention a lot of unstable Z-odd/N-odd nuclei exist)

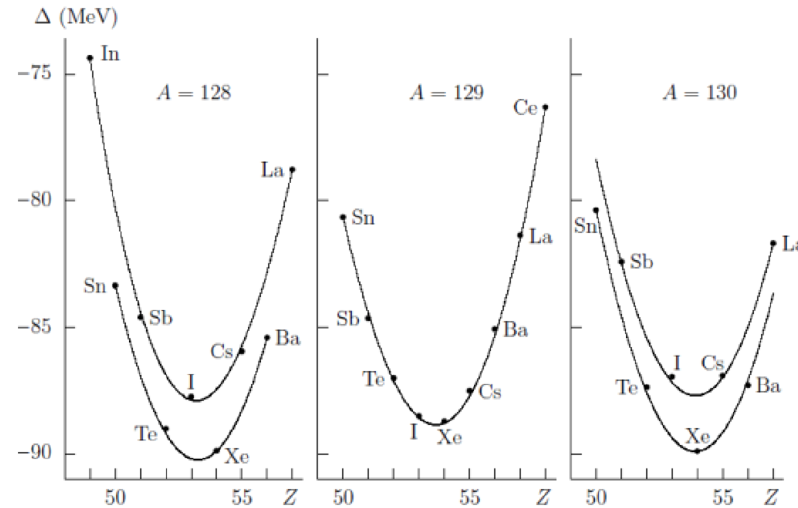
# Bethe-Weizsäcker formula (3)




- The observed binding energies as a function of *A* and the predictions of the mass formula
- Only even–odd combinations of *N* and *Z* are considered → pairing term vanishes

# Bethe-Weizsäcker formula (4)

- The Bethe-Weizsäcker formula explains the parabolic behaviour for the masses → for *A* = constant → second order polynomial in *Z* → stability valley




- The parabola is centered about the point where the equation

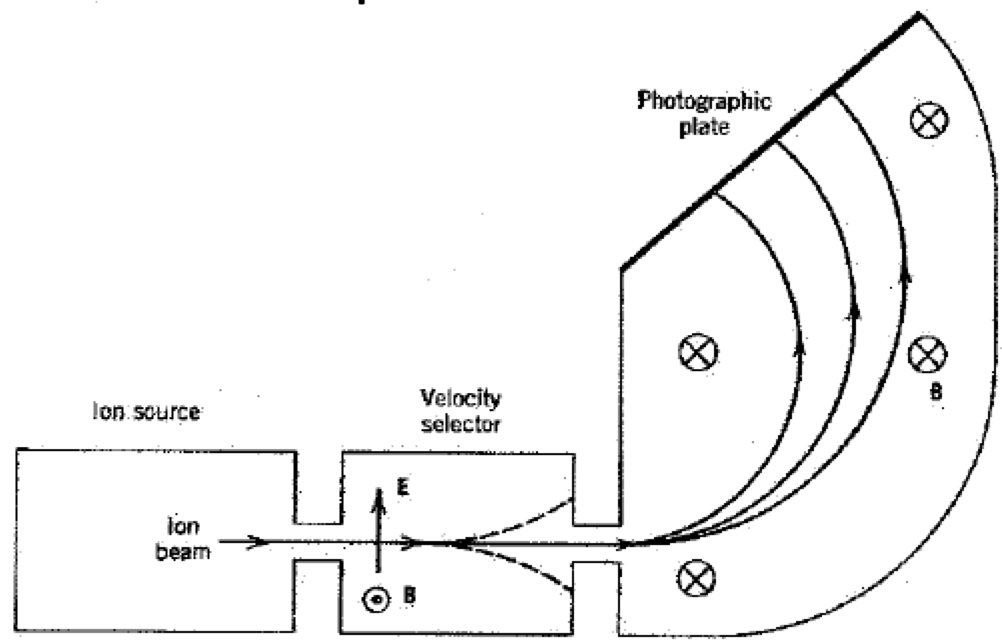
$$m(A,Z) \simeq N m_n + Z M(^1H) - B(A,Z)/c^2$$

reaches the minimum ∂M/∂Z = 0 →

$$Z_{min} = \frac{[m_N - m(^1H)] + a_C A^{-1/3} + 4a_a}{2a_C A^{-1/3} + 8a_a A^{-1}} \simeq \frac{A}{2}\frac{1}{1+0.0078A^{2/3}}$$

- The splitting for even-*A* is due to pairing acting in opposite directions for even-even nuclei (lower parabola) and odd-odd nuclei (upper parabola)

# Mass spectrometer




- Production of an ion beam with thermal distribution of velocities
- A selector passes only ions with a particular velocity $v$
- Momentum selection by magnetic field $B$ permits mass identification → $r = mv/qB$

# Nuclei stability: Stability in particles

- Different notions of stability
- First definition: **stable in particles** → no possible dissociation in sub-systems with smaller total energy
- We consider $(A,Z) \rightarrow (A_1,Z_1) + (A_2,Z_2)$ with $A = A_1 + A_2$ and $Z = Z_1 + Z_2$ → stable if ($\forall$ $A_1$, $Z_1$):

$$m(A,Z) < m(A_1,Z_1) + m(A_2,Z_2) \text{ or } B(A,Z) > B(A_1,Z_1) + B(A_2,Z_2)$$

- The nuclear mass can be replaced by the atomic mass except in some cases where the stability depends on the presence of the $e^-$
- This case corresponds to a spontaneous fission, to the emission of $\alpha$, neutron, proton,... → **instability in particles** → $A$ changes → lifetime generally very short $\sim 10^{-21}$ s

# Nuclei stability: Absolute stability

- More generally → a nucleus is **absolutely stable** if ($\forall$ $m_i$):

$$m(A, Z) < \sum_i m_i$$

- The sum concerns all possible masses and all possible disintegration modes
- If absolute stability → stability in particles

# Nuclei stability: Instability by $\beta$ emission

- Attention → the previous definitions are not sufficient

- Other instabilities exist due to the weak force ($\beta$ disintegration) → **instability by $\beta$ emission**

- If $\beta$ emission → *A* is constant but *N* and *Z* change

- Variable lifetime from $10^{-6}$ s to $10^{15}$ years

# Nuclei stability: Examples (1)

- $^{12}C$: no possible splitting emitting energy from ground state, no $\beta$ emission → stable
- $^{8}Be$: can split into 2 $^{4}He$ → $M(8,4) - 2M(4,2) \approx 0.092$ MeV → unstable (lifetime ≈ $10^{-16}$ s)
- $^{3}H$: stable in particles but by $\beta$ disintegration → $^{3}He$ (lifetime ≈ 12.3 years)
- Information may be deduced from →

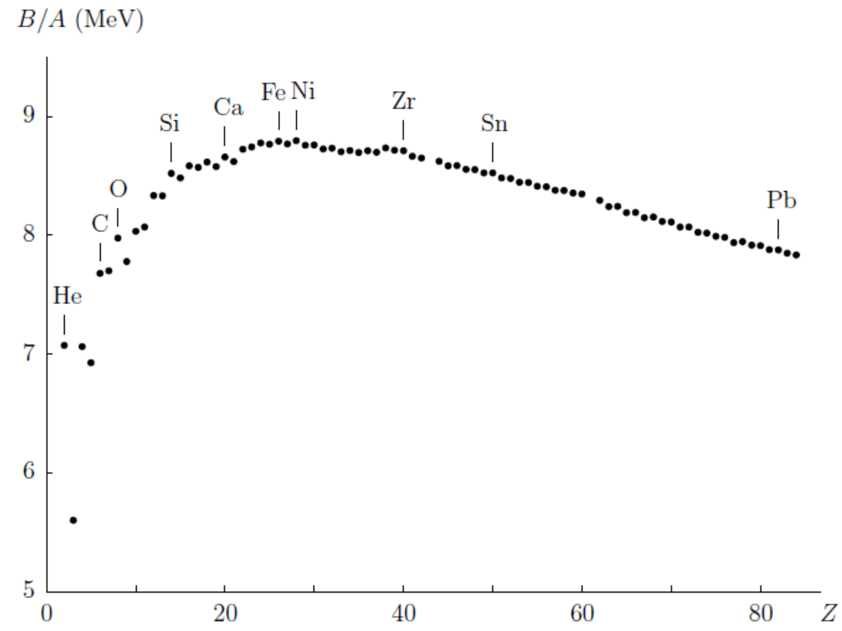
$$B(A,Z) \quad > \quad B(A_1,Z_1) + B(A_2,Z_2)$$

$$\Leftrightarrow$$

$$A_1\left(\frac{B(A,Z)}{A} - \frac{B(A_1,Z_1)}{A_1}\right) \quad + \quad A_2\left(\frac{B(A,Z)}{A} - \frac{B(A_2,Z_2)}{A_2}\right) > 0$$

# Nuclei stability: Examples (2)

- This relation is fulfilled for nucleus corresponding to the maximum (Fe) of the curve and for nuclei at the left of the maximum ↔ their fragments have smaller *B/A* ratios → some nuclei are certainly stable
- For heavy nuclei → completely ≠ → they are beyond the maximum → they provide energy during splitting → the 2 terms of previous equation are negative



# Nuclei stability: Examples (3)

- $^{208}Pb$: various dissociations seems possible: 2 examples:

$$^{208}Pb \to \alpha + {}^{204}Hg + 0.52 \text{ MeV}$$

$$^{208}Pb \to 2^{104}Zr + 110 \text{ MeV}$$

- Practically → none of these processes is observed and $^{208}Pb$ is stable (only one heavier nucleus is stable: $^{209}Bi$)
- The lifetime of $^{208}Pb$ is so long → disintegrations can be considered as negligible → stable
- One of the reasons for this long lifetime is that the system of nucleons has to cross the potential barrier to split up → the probability of crossing may be so small that the lifetime is extremely long

# Nuclei stability: Conventional stability

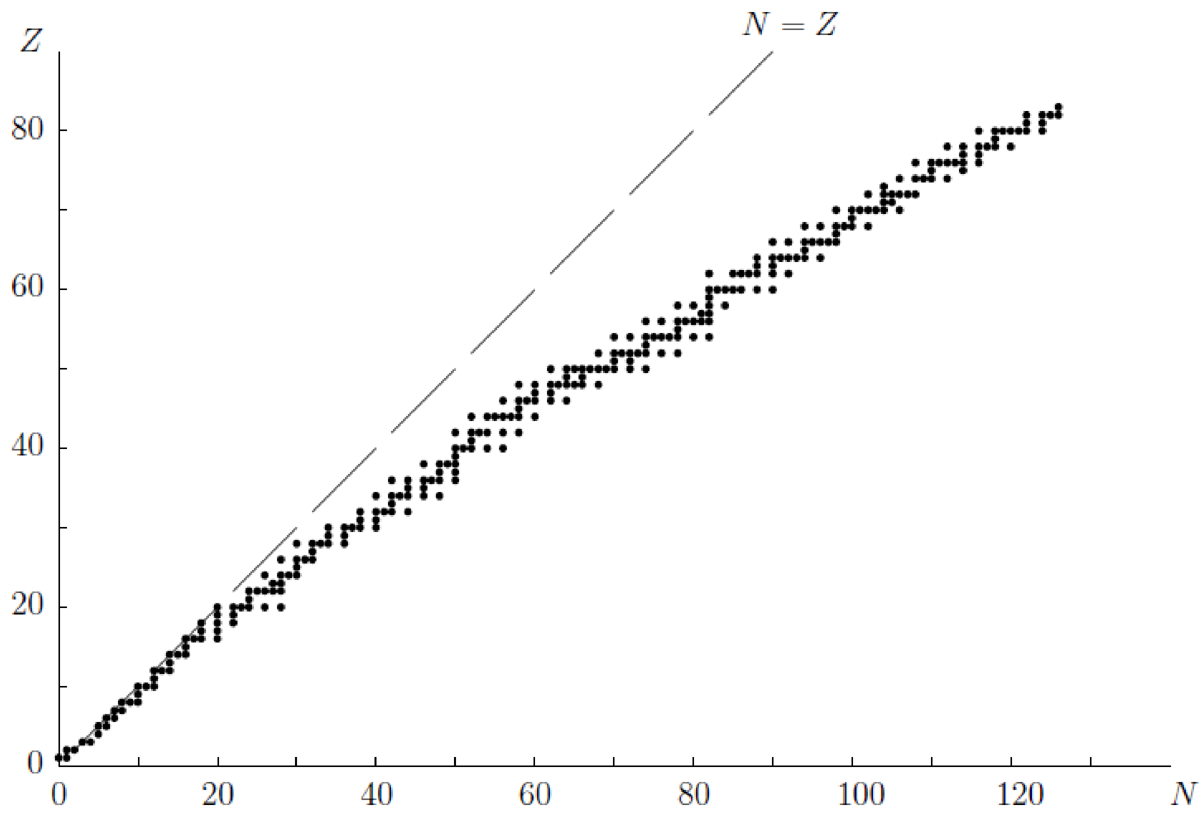
- Finally the convention is to consider that a nucleus is **stable if its lifetime is larger than the age of Universe** →

$$\tau \gg 1.5 \times 10^{10} \text{ years} \simeq 5 \times 10^{17} \text{ s}$$

- Practical definition even though it is artificial
- Example: $^{209}Bi$ has $\tau$ = 1.9 $10^{19}$ years → stable but disintegration was observed:

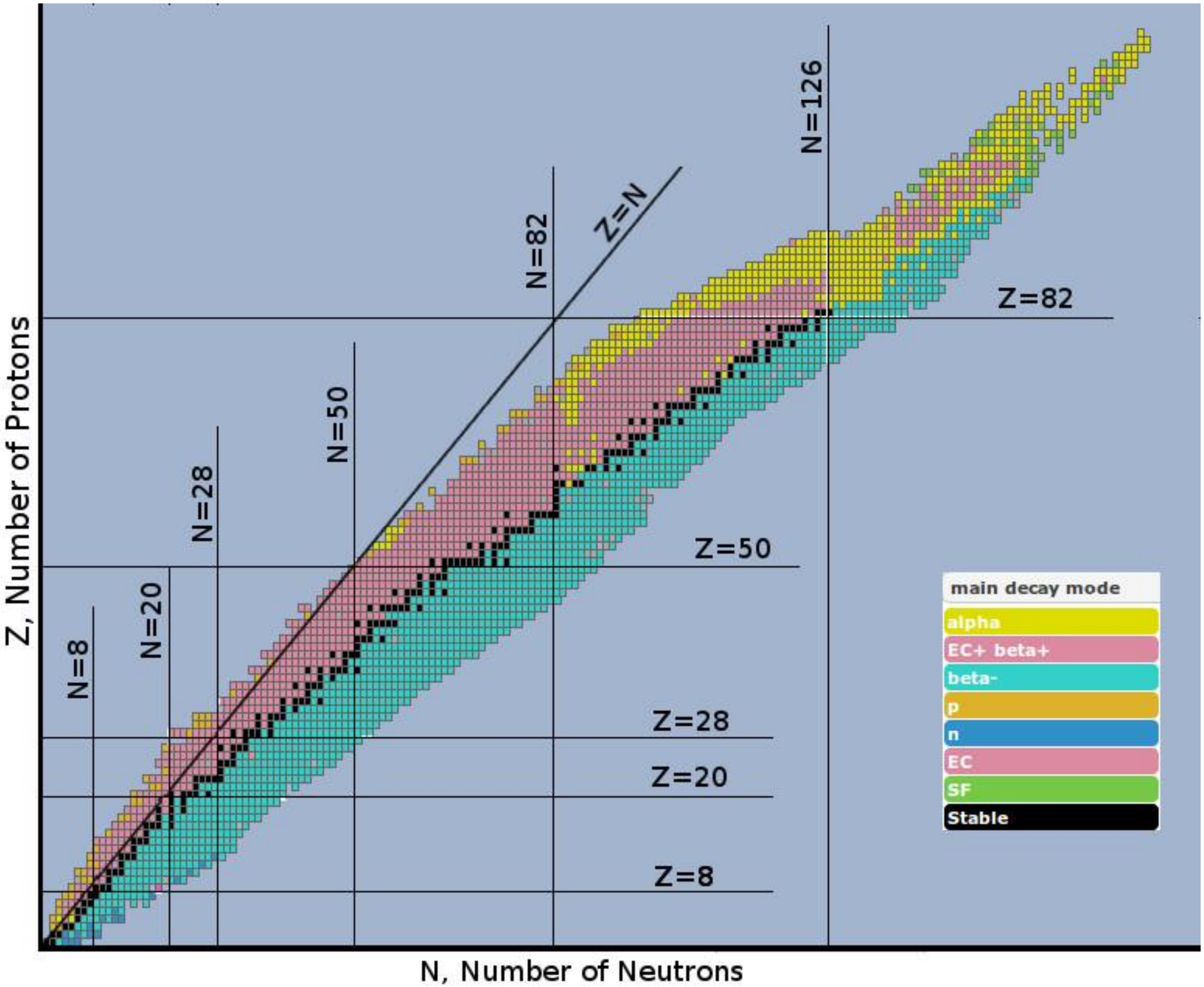
$$^{209}Bi \rightarrow \alpha + {}^{205}Tl$$

# Stable nuclei (1)

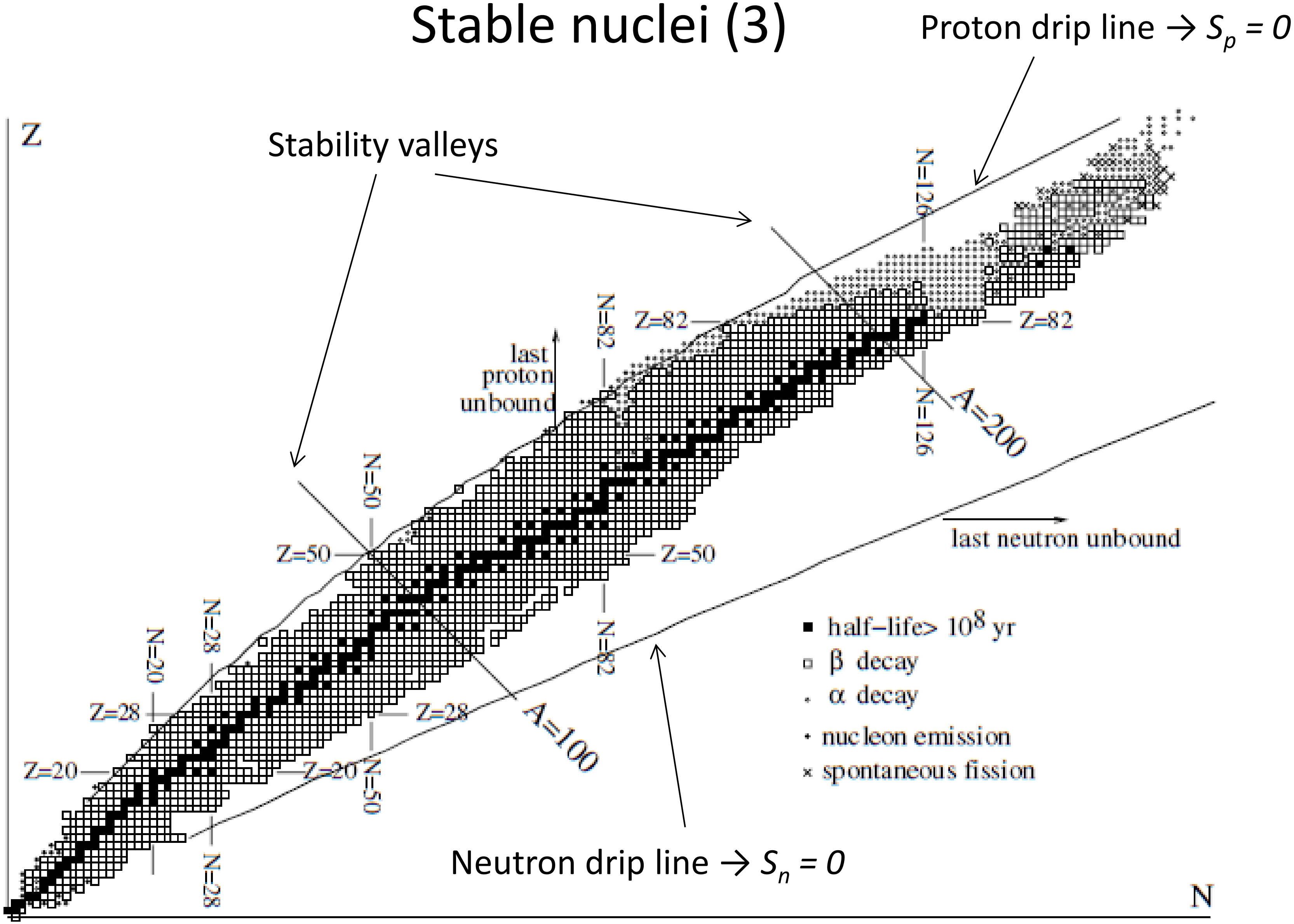



- For $N$ and $Z < 20 \rightarrow$ stable nuclei close to the straight line $N = Z$ (only $^{3}He$ is upper)
- Tc ($Z = 43$) and Pm ($Z = 61$) have no stable isotope
- For $N$ and $Z > 20 \rightarrow$ stable isotopes move away from $N = Z$ line $\rightarrow$ increasing effect of Coulomb repulsion $\rightarrow$ for last stable nuclei $N/Z = 1.5$

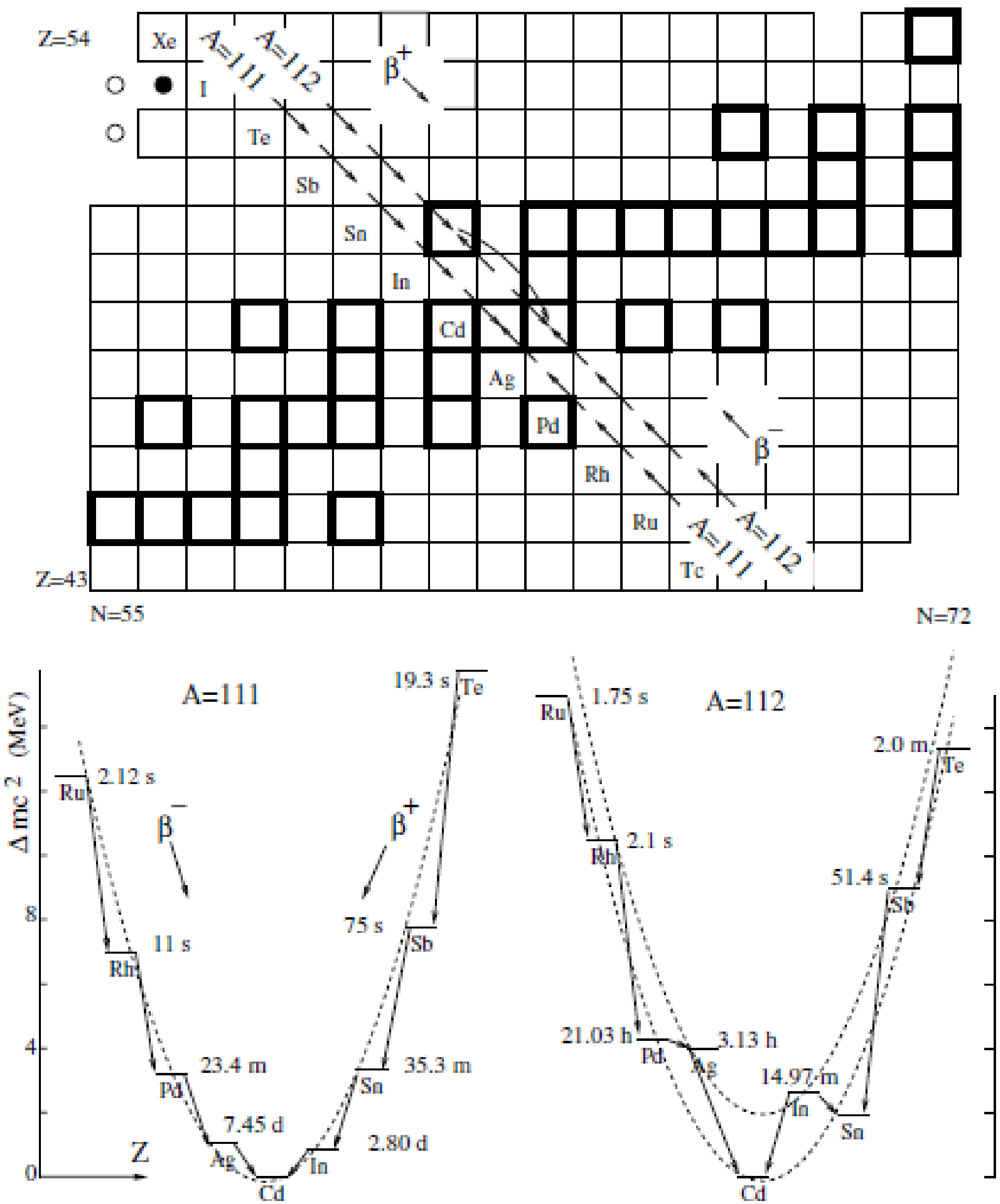
# Stable nuclei (2)



# Stable nuclei (3)



# Stable nuclei (4)



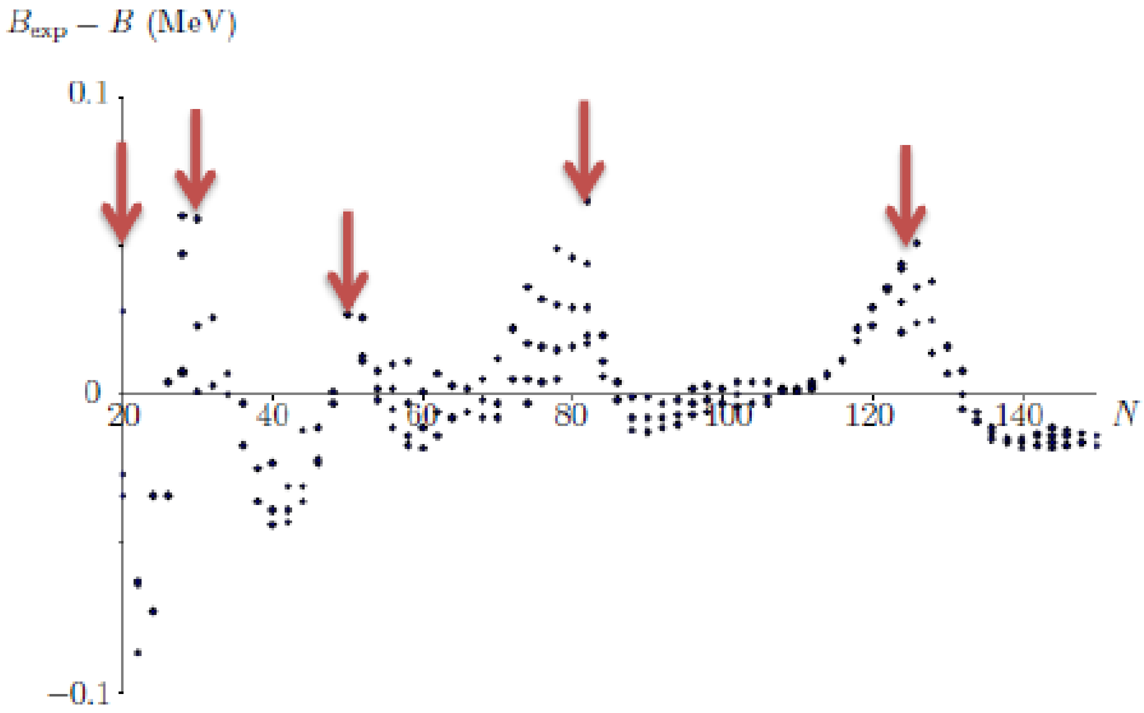

- Nuclei with an excess of neutrons (below the $\beta$ stable nucleus) decay via $\beta^-$ emission
- Nuclei with an excess of protons (above the $\beta$ stable nucleus) decay via $\beta^+$ emission or electron capture
- The dashed lines show the predictions of the Bethe-Weizsäcker formula
- Note that for even-*A* → two stable isobars $^{112}$Sn and $^{112}$Cd

# Magic numbers (1)

- In atomic physics → the ionization energy *I* (the energy needed to extract an electron from a neutral atom) shows discontinuities around $Z$ = 2, 10, 18, 36, 54 and 86 (i.e. for noble gases) → these discontinuities are associated with closed electron shells
- An analogous phenomenon occurs in nuclear physics → there exist many experimental indications showing that atomic nuclei possess a shell-structure → they can be constructed (like atoms) by filling successive shells of an effective potential well (shell model)
- Separation energies present discontinuities at special values of $N$ or $Z$ → these numbers are called *magic numbers*
- These numbers are 2 – 8 – 20 – 28 – 50 – 82 – 126
- The discontinuity in the separation energies is due to the excess binding energy for magic nuclei as compared to that predicted by the Bethe-Weizsäcker formula

# Magic numbers (2)

- For the same reason (shell effect not considered in the liquid drop model) → the difference between the experimental binding energy $B_{exp}$ and the binding energy $B$ calculated from the Bethe-Weizsäcker formula is maximum for these $N$ values

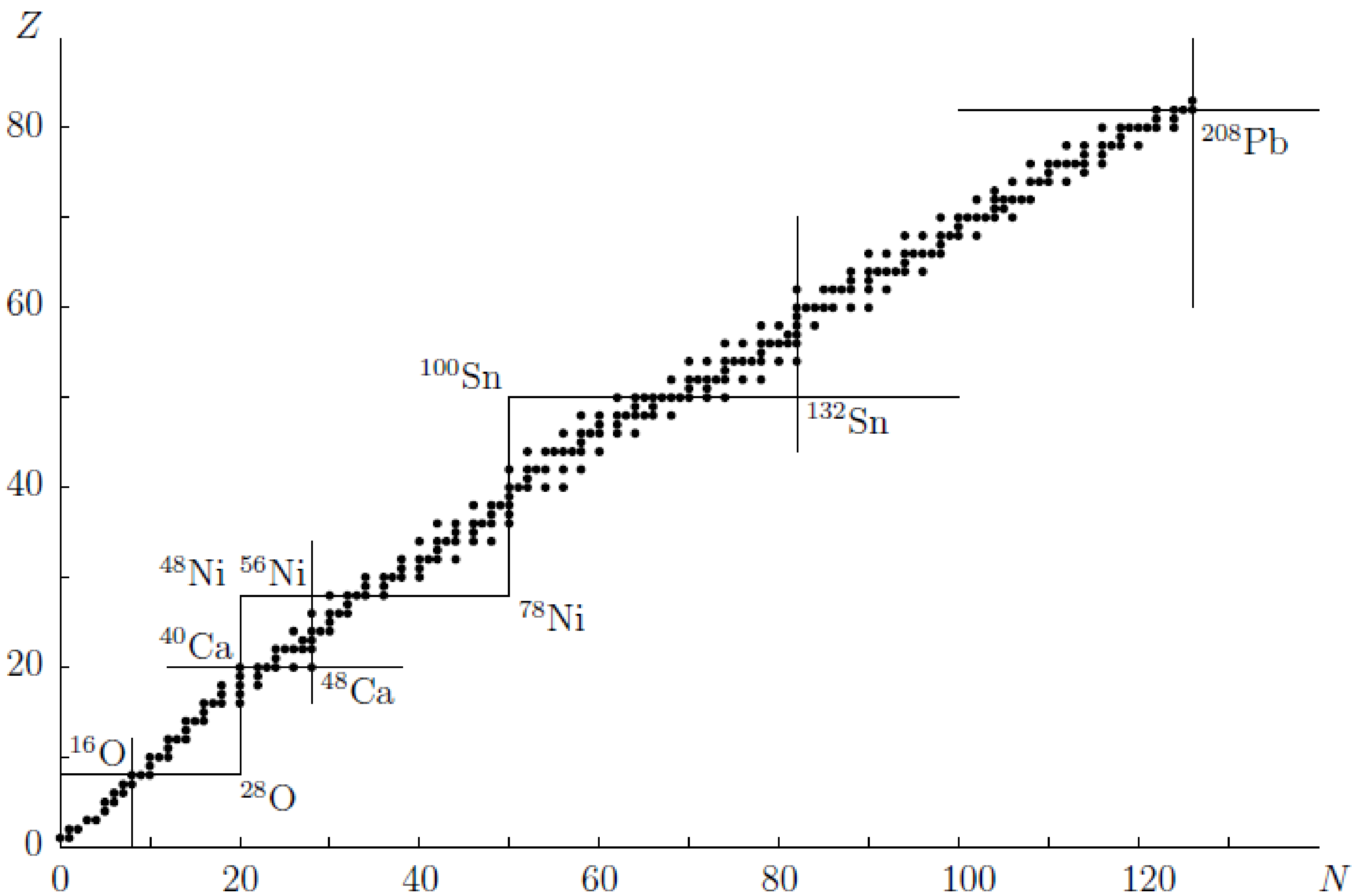



This difference is also observed as a function of $Z$

# Magic numbers (3)

- Equivalent variations or discontinuities appear for other quantities as radius, electric and magnetic moments,…
- Nuclei with magic numbers of neutrons or protons have a closed shell that encourages a spherical shape
- Nuclei having both magic neutrons and protons are particularly stable → they are called *doubly magic* nuclei → $^{4}$He ($Z$ = 2), $^{16}$O ($Z$ = 8), $^{40}$Ca ($Z$ = 20), $^{208}$Pb ($Z$ = 82)
- Some nuclei having magic number can be instable → but generally with a radioactivity smaller than waited → $^{28}$O ($Z$ = 8), $^{48}$Ni, $^{56}$Ni, $^{78}$Ni ($Z$ = 28), $^{100}$Sn, $^{132}$Sn ($Z$ = 50)
- The following magic number (not observed) could be 184

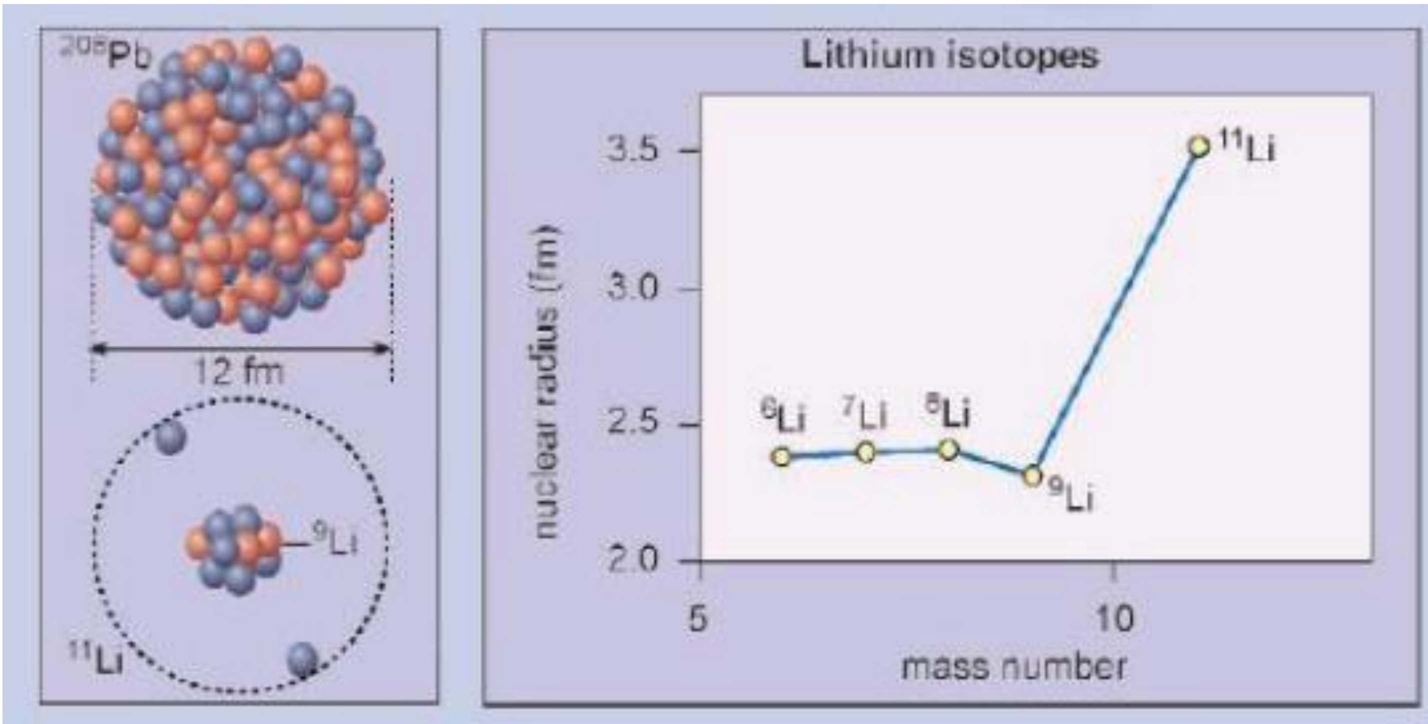
# Magic numbers (4)



# Particular types of nuclei (1)

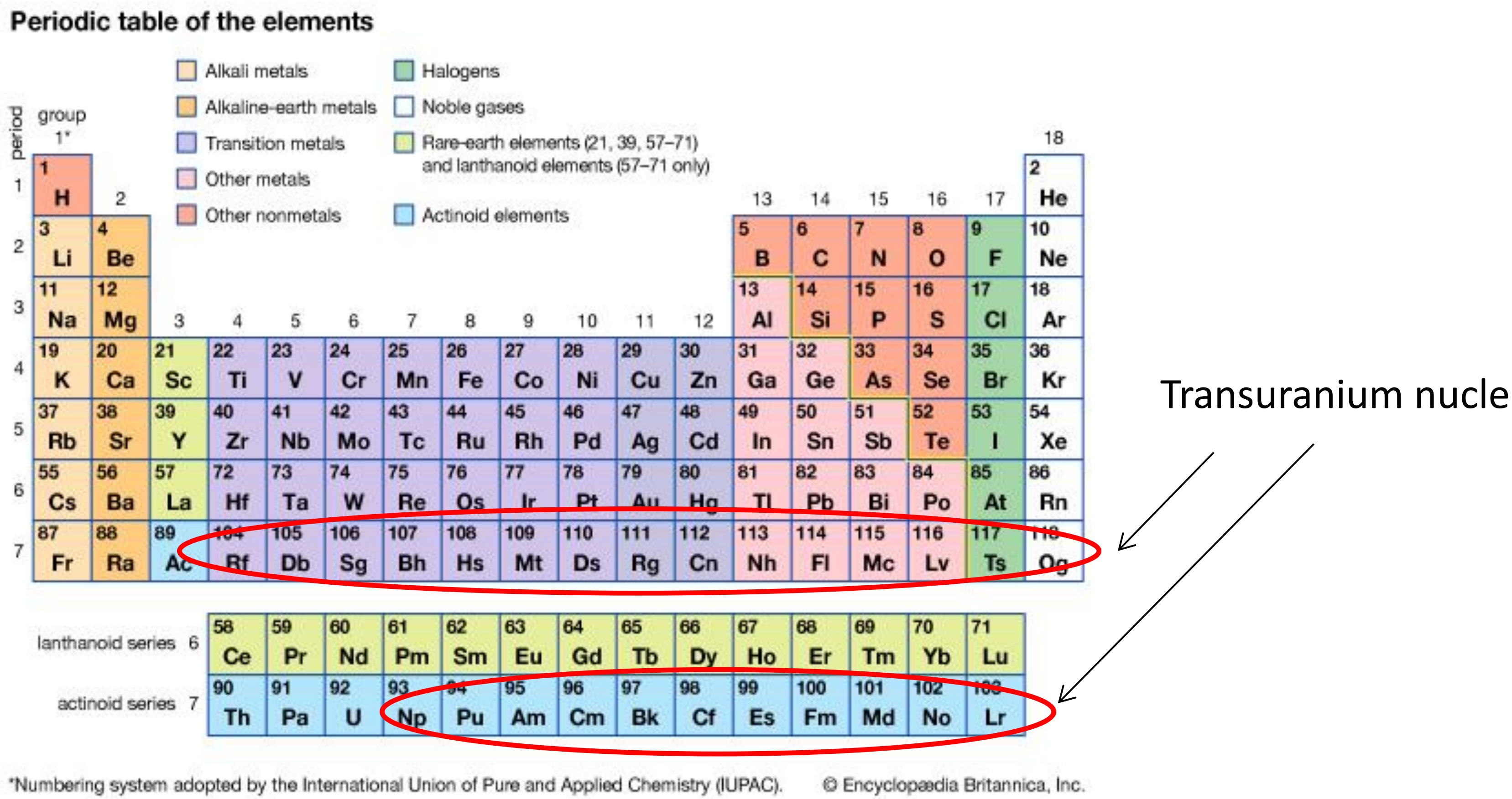
- Exotic nuclei: Instable nuclei characterized by a number of neutrons or protons vary far from the stability valley (examples: $^{24}O$,…)
- Halo nuclei: Exotic nuclei with a radius appreciably larger than that predicted by the rule $R = R_0 A^{-1/3}$ → they are characterized by a core nucleus (with normal radius) surrounded by a halo of orbiting protons or neutrons → they have necessarily very weak separation energy (examples: $^{8}He$, $^{22}C$ …)
- Transuranium nuclei (also called transuranic nuclei) nuclei with atomic number greater than 92 ($Z$ of uranium, last natural element):

# Particular types of nuclei (2)




Example of halo nuclei: $^{11}$Li

# Particular types of nuclei (3)




- Superheavy nuclei: Hypothetic nuclei with life time larger than the last transuranium nuclei due to the proximity of the next magic number (126) → island of stability