Chapter III:
General properties of nuclei

Summary

1. General information
2. Notations, quantum numbers and spectra
3. Nuclear radius
4. Nuclear electromagnetic moments
5. Nuclear and atomic masses
6. Nuclei stability
7. Magic numbers
8. Particular types of nuclei

Units and dimensions: Typical values

- Size of the atom ≈ 10^{-10} m
- Size of the nucleus ≈ 10^{-15} m = 1 femtometer (fm) = fermi → all nuclei have radius = 2-8 fm
- Typical β or γ decay energy in the range of 1 MeV (megaelectron-volt) = 10^6 eV = 1.6021765 $\times$ 10^{-13} J (1 eV = energy gained by a single unit of charge when accelerated through a potential difference of 1 V)
- Unit of mass → 1 unified atomic mass unit (u) = 1.6605390 $\times$ 10^{-27} kg → 1/12 of the mass of an unbound neutral atom of ^{12}C (in ground state and at rest)
- Practically → use of mass energy rather than mass → multiplication by c^2 (c = 299 792 485 ms^{-1} ≈ 3 $\times$ 10^8 ms^{-1}) → 1 u = 931.502 MeV
- Unit of charge → elementary charge (e) = 1.6021766209 C (proton: e, electron: -e)
- Mean lifetime τ = 1/λ with λ = probability of disintegration per unit time → can be 10^{-21} or 10^{19} s

Units and dimensions: Multiplication by a power of c

- Mass m → mc^2 (energy)
- Momentum p → pc (energy)
- Time t → tc (length)

- Physical constants:
  - Planck constant: $\hbar = 1.05 \times 10^{-34}$ Js → $\hbar c = 197.33$ MeVfm
  - Proton mass: $m_p = 1.6726 \times 10^{-27}$ kg → $m_p = 938.27$ MeV/c^2
  - Neutron mass: $m_n = 1.6749 \times 10^{-27}$ kg → $m_n = 939.57$ MeV/c^2
  - Electron mass: $m_e = 9.1094 \times 10^{-31}$ kg → $m_e = 0.511$ MeV/$c^2 \approx m_p/1836$

Structure of nuclei: Nucleons

- Atomic nuclei are quantum bound states of particles called *nucleons*
- Two types of nucleons → positively charged proton and uncharged neutron
- The mass difference between proton and neutron is known with a huge precision: $m_n - m_p = 1.293\ 332\ \mathrm{MeV}/c^2$
- Nucleons are fermions (spin ½)
- Nucleon is not an elementary particle → an elementary particle has its root mean square (rms) radius = 0

Root mean square radius of proton and neutron (1)

- The rms radius (r_{rms}) of a particle (or charge radius) is defined as the radius of the charge distribution inside the particle →

$$\langle r^2 \rangle_{ch}^{1/2} = \sqrt{\int r^2 \rho_{ch}(\boldsymbol{r}) d\boldsymbol{r}}$$

For a proton →

$$\langle r^2 \rangle_{ch,p}^{1/2} \approx 0.87 \text{ fm}$$

with

$$\int \rho_{ch,p}(\boldsymbol{r}) d\boldsymbol{r} = 1$$

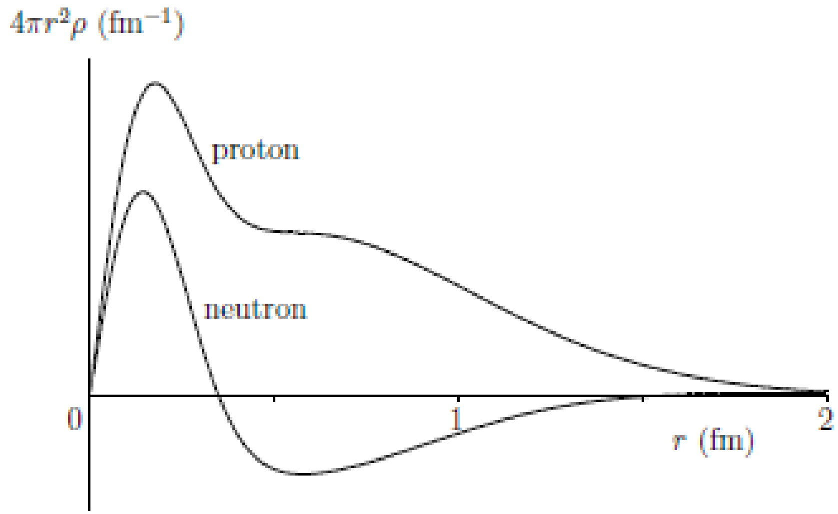
For a neutron →

$$\langle r^2 \rangle_{ch,n} \approx -0.12 \text{ fm}^2$$

with

$$\int \rho_{ch,n}(\boldsymbol{r}) d\boldsymbol{r} = 0$$

Root mean square radius of proton and neutron (2)



- The charge density of a nucleon is measured from the analysis of high energy electrons scattered from it
- Practically → measurement of charge density is made through their Fourier transform *F(**q**)* such as

$$F(\boldsymbol{q}) = \frac{1}{Z}\int e^{i\boldsymbol{q}\boldsymbol{r}}\rho(\boldsymbol{r})d\boldsymbol{r}$$

Structure of nuclei: Magnetic moment

- The spin magnetic moment is the magnetic moment induced by the spin of elementary particles:

$$\boldsymbol{M} = g\frac{q}{2m}\boldsymbol{S}$$

with ***S***: the spin, *q*: the charge, *m*: the mass, *g*: the gyromagnetic ratio

- From Dirac theory → for charged fermion: *g = 2*, for neutral fermion: *g = 0* (small corrections from quantum electrodynamics - QED)
- Nucleon is not an elementary particle → g_p = 5.5856947 and g_n = -3.826085

Structure of nuclei: Quarks (1)

- Nucleons are composed of 3 elementary particles: the quarks

Symbol	Spin	Charge	Flavor
u	1/2	+2/3	Up
d	1/2	-1/3	Down
c	1/2	+2/3	Charm
s	1/2	-1/3	Strange
t	1/2	+2/3	Top
b	1/2	-1/3	Bottom

+ anti-quarks

- Quarks have supplementary quantum number: color charge (red, blue, green)
- Anti-quarks have anticolor (antired, antiblue, antigreen)
- Due to the phenomenon of *color confinement* → quarks are never directly observed or found in isolation
- Quarks combine to form hadrons

Structure of nuclei: Quarks (2)

- Theory to be applied to systems of quarks → quantum chromodynamics (QCD)
- Combination of 2 quarks (quark + anti-quark) → meson (pion, kaon,…)
- Combination of 3 quarks with different colors → baryon (proton, neutron, hyperon,…)
- Formation of tetraquarks and pentaquarks seems possible (seems to be observed at the CERN)
- Proton → p = u + u + d
- Neutron → n = u + d + d
- Spin of the nucleon (1/2) results from the coupling of 3 spins 1/2

Lepton

- Lepton is an elementary particle
- Spin = 1/2
- Examples:
  - Electron: charge e, mass ≈ 0.5109989 MeV/c^2 ≈ 511 keV/c^2
  - Positron: charge -e, mass ≈ 511 keV/c^2
  - Muon (« heavy electron »): charge –e, $m_\mu \approx 209\, m_{e-}$)
  - Neutrino: charge 0, mass ≈ 0 but not 0 → $m_\nu < 3\ eV/c^2$, 3 flavors (electron, muon, tau), negative helicity (projection of spin onto the direction of momentum)
  - Antineutrino: same charge and mass than neutrino → really different? → not clear but all experiments have shown positive helicity

Types of forces

Force	Amplitude	Range
Strong nuclear interaction	$\sim$ 1	$\sim$ fm
Coulomb interaction (or electromagnetic)	$\sim$ 1/137	Infinite
Weak nuclear interaction	$\sim 10^{-5}$	$\sim 10^{-3}$ fm
Gravitation interaction	$\sim 10^{-39}$	infinite

- Gravitational interaction is negligible compared to the other ones → but for systems with a huge number of particles → becomes dominating because of the weak total charge of macroscopic systems
- In nuclei → Coulomb interaction is not negligible → becomes very important when the number of protons ↗

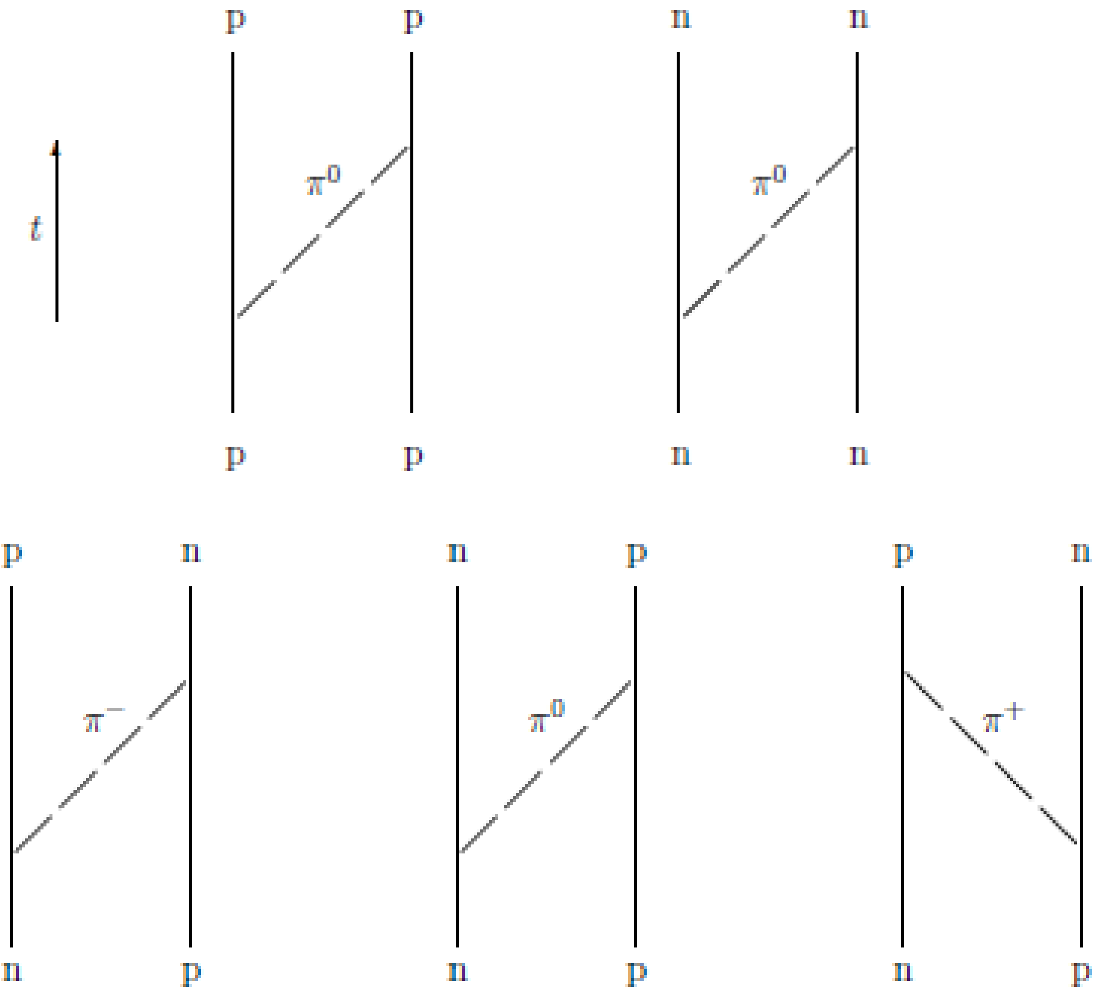
Strong nuclear interaction (1)

- During interactions → exchange of virtual particles between particles (Coulomb interaction → exchange of photons)
- For strong nuclear interaction between nucleons → exchange of mass virtual particles (Yukawa theory) → pions (or pi mesons) Π^+, Π^- and Π^0 (index = charge)
- Due to time-energy uncertainty relation → $\Delta t \Delta E \simeq \hbar$
- The energy fluctuation ΔE necessary to have a possible reaction is $\Delta E \sim m_\Pi c^2$
- In the time interval Δt the pion car travel a distance $\sim c\Delta t \sim \hbar/(m_\Pi c)$ with $\hbar$/mc the reduced Compton wavelength
- This distance gives the range of the nuclear force

$$m_{\Pi^+} \simeq 139.570\ \mathrm{MeV/c^2} \quad \Rightarrow \quad \lambda_{\Pi^+} \simeq 1.414\ \mathrm{fm}$$

$$m_{\Pi^0} \simeq 134.977\ \mathrm{MeV/c^2} \quad \Rightarrow \quad \lambda_{\Pi^0} \simeq 1.462\ \mathrm{fm}$$

Strong nuclear interaction (2)



Feynmann diagrams for n/p interactions

Strong nuclear interaction (3)

- Same physical mechanism for interaction between 2 protons, 2 neutrons and 1 proton/1 neutron → exchange of same type particle → very similar interactions → property of *charge independence*
- The strength of the strong interaction between any pair of nucleons is the same independently of the nucleon type (protons or neutrons)
- Charge independence is not perfect because Π^0 is necessary exchanged for nn and pp interactions and np interactions can be done with various Π → as potentials are not exactly the same → not perfect charge independence

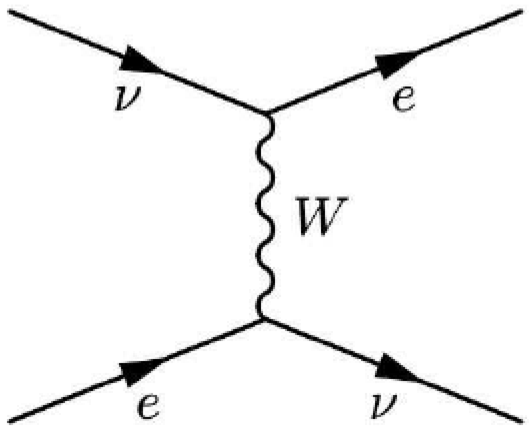
Weak nuclear interaction (1)

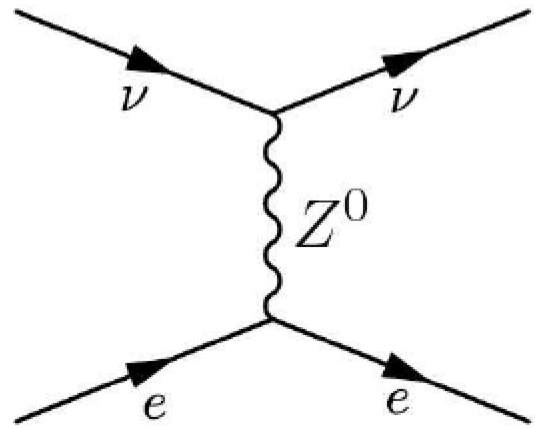
- Weak nuclear interaction always neglected in all calculations of nuclear structure
- This effect only appears in processes forbidden to strong nuclear interactions → β disintegration
- Yukawa theory can be adapted to weak interaction → weak gauge bosons (W^+, W^-, Z^0) mediate the weak interactions

$$m_W \simeq 80.4 \text{ GeV/c}^2 \quad \Rightarrow \quad \lambda_W \simeq 0.00246 \text{ fm}$$
$$m_Z \simeq 91.19 \text{ GeV/c}^2 \quad \Rightarrow \quad \lambda_Z \simeq 0.00217 \text{ fm}$$

- Short range → weakness of the interaction

Weak nuclear interaction (2)





- The electroweak theory unifies weak interaction and electromagnetic interaction
- This theory shows that at high energy ($E > m_Z c^2$) → the 2 interactions have the same order of magnitude

Conservation laws (1)

- A *conservation law* states that a particular physical quantity does not changed (is conserved) during a physical process
- A conservation law is *exact* if it was never contradicted
- A conservation law *is approximated* if the considered quantity is conserved in certain classes of physics processes but not in all (e.g. conservation of parity)
- Principals conservation laws are:
  - conservation of energy and momentum (exact)
  - conservation of angular momentum (exact)
  - conservation of electric charge (exact)
  - conservation of baryon number (approximated)
  - conservation of lepton number (approximated)
  - …

Conservation laws (2)

- Conservation of baryon number → conservation of the total number of nucleons (neutrons and protons) minus the total number of anti-nucleons (antiprotons and antineutrons) → p + d $\nrightarrow$ p + p + γ is not allowed (charge YES, baryon number No) → γ + d $\rightarrow$ p + n is allowed
- Conservation of electronic lepton number (L_e) → conservation of the difference between the total number of [electrons and electron neutrinos] and the total number of [positrons and electronic antineutrinos] →

$$L_e \equiv N(e^-) + N(\nu_e) - N(e^+) - N(\bar{\nu}_e)$$

→ $\bar{\nu}_e$ + n $\nrightarrow$ e^- + p is NOT allowed but → $\bar{\nu}_e$ + n $\rightarrow$ e^+ + n YES

- Remark 1: it exists two other types of charged leptons → $\mu^\pm$ and $\tau^\pm$ → similar conservation laws with L_μ and L_τ

Conservation laws (3)

- Remark 2: some recent experiments on neutrino oscillations shows that the only truly conserved number is the sum of the 3 lepton numbers: L = $L_e + L_\mu + L_\tau$

- Remark 3: p $\nrightarrow$ $e^+ + \gamma$ is not possible (conservation of baryon and lepton number) but a « possible » theory predicts the disintegration of proton (with a characteristic lifetime $\tau_p \approx 10^{29}$ years ≈ 10^{19} times the age of Universe)

- Remark 4: neutron is instable → n $\rightarrow$ p + e^- + $\overline{\nu}_e$ (with $Q = (m_n - m_p - m_e)^2 \approx 0.782$ MeV and $\tau_n \approx 885.7$ s)

Conservation laws: Stability of nuclei

- Conservation of energy → 1 particle of mass *m* and charge *q* can spontaneously decay into *i* particles only if (with *Q* the liberated energy):

$$m > \sum_i m_i \text{ or equivalently} \rightarrow Q = mc^2 - \sum_i m_i c^2 > 0$$

- Conservation of charge → $q = \sum_i q_i$

- These laws implies that electron and positron are stable (no particle of same charge but of smaller mass)

Origin of nuclei

- Nuclei in nature were built by nuclear reactions since the « Big Bang »
- Begin not well known → system at very high T → free quarks and gluons (bosons mediating between quarks) = quark-gluon plasma
- Time t ↗ → T ↘ → quarks and gluons combine to form hadrons and nucleons → proton $\rightarrow$ neutron and neutron $\rightarrow$ proton → but as $m_p < m_n$ → system with smallest mass is favored
- t ↗ more → T ↘ → collisions → apparition of bounded systems (with A $\leq$7) → fixed situation = 87% of protons and 13% neutrons
- Abundance: 74% H + 23-25% He (4p $\rightarrow$ ^{4}He + 2e$^+$ + 2ν_e+ nγ) + …
- Other nuclei are formed during star explosion: supernova → a lot of nuclei are unstable and decay into other nuclei → process of formation + decay is called *nucleosynthesis*

Notations: Nuclide

- A nuclear species – *nuclide* – is defined by the number of neutrons *N* and by the number of protons *Z* (called the atomic number → charge in unit of *e*) → the mass number *A* is the total number of nucleons (the integer closest to the mass of the nucleus in *u*) → *A* = *N* + *Z* → with *X* the chemical symbol →

$$(A, Z) \leftrightarrow {}^{A}_{Z}X_{N} \leftrightarrow {}^{A}_{Z}X \leftrightarrow {}^{A}X$$

- Example: deuteron → $(2, 1) \leftrightarrow {}^{2}_{1}H_{1} \leftrightarrow {}^{2}_{1}H \leftrightarrow {}^{2}H$
- *Isotopes* → have same charge *Z* but different *N*: ${}^{235}_{92}\mathrm{U}$ and ${}^{238}_{92}\mathrm{U}$
- *Isobars* → have the same mass number *A* → ${}^{3}_{2}\mathrm{He}$ and ${}^{3}_{1}\mathrm{H}$
- *isotone* → have the same *N* but different *Z* → ${}^{14}_{6}\mathrm{C}_{8}$ and ${}^{16}_{8}\mathrm{O}_{8}$

Quantum numbers

- (A,Z) define a nuclear species → not the nuclear *quantum state*
- In atoms → individual electrons can move to higher energy orbits ↔ in nuclei → same for individual nucleons
- Nucleus (*A,Z*) has a rich spectrum of excited states (with few exceptions) which can decay to the ground state by emitting photons (γ-rays)
- Energy levels of a nucleus (including ground state) are characterized by *good quantum numbers* (integers or half-integers) corresponding to eigenvalues of operators (called *constants of motions*) commuting with the Hamiltonian *H* of the nucleus
- Constants of motions are deduced from symmetries of dominating interactions of nucleus (strong + Coulomb)

Good quantum numbers

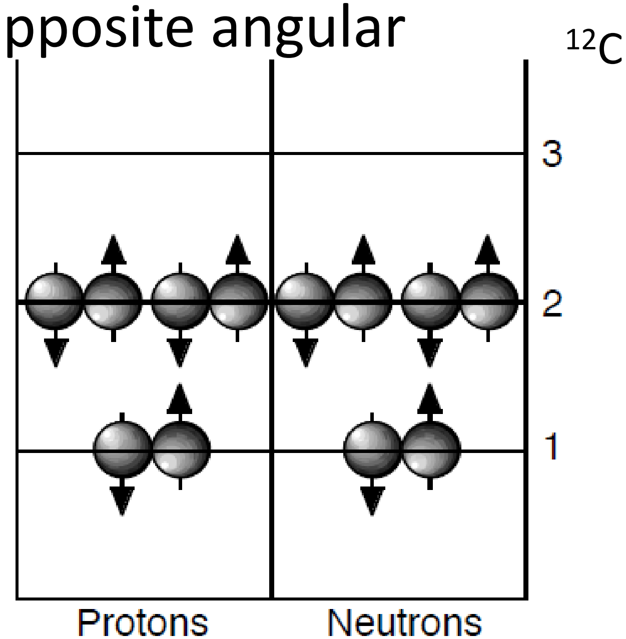
- Invariance under rotation → total angular momentum ***J*** = constant of motion
- Invariance under reflection (sometimes violated but weak violation in nuclei) → parity Π = constant of motion
- Other good quantum number → total angular momentum projection J_z
- Complete set of commuting observables → {H, ***J***, J_z, Π}
- Nuclear levels are noted →

$$J^{\pi} \leftrightarrow J^{\pi}, E_x$$

- J is the total angular momentum quantum number, π is the parity quantum number and E_x is the excitation energy compared to the ground state
- Remark: E_x does not depend on the quantum number M associated to J_z (different energy states for same E_x and J)

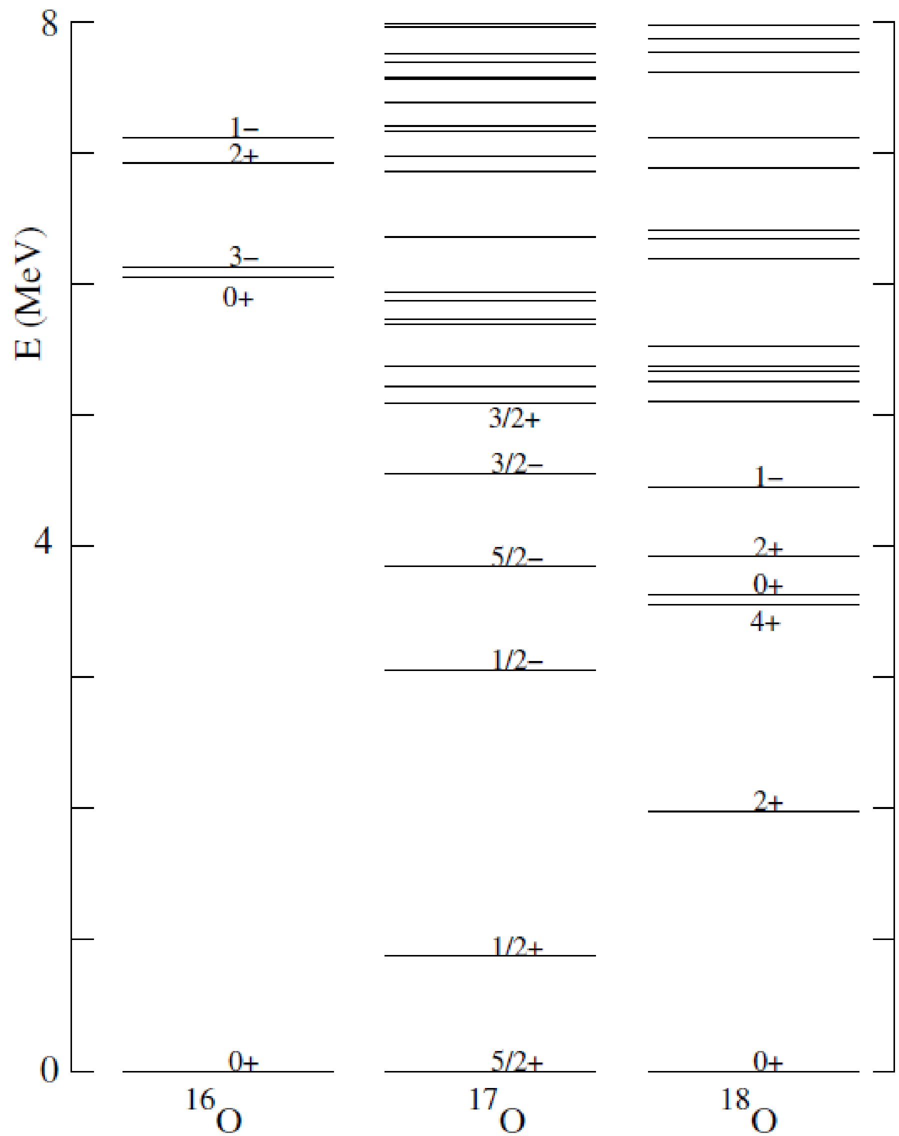
Quantum numbers: Ground state (1)

- Among all states of the nuclei → the most important is the ground state → some simple rules exist to determine its quantum state
- To obtain ground state → fill nucleons in lowest energy first
- To obtain ground state → pair up nucleons as you add them (« Katz's rule »)
- The ground state of all *N*-even and *Z*-even stable nuclei is characterized by the quantum numbers 0^+ ↔ identical nucleons tends to pair with another nucleon of the opposite angular momentum → $J = 0$
- The parity is a statement about what the nuclear structure of the state would look like if the spatial coordinates of all the nucleons were reversed → π = + means the reversed state = the original ↔ if even-even nucleus → π = +



Quantum numbers: Ground state (2)

- The ground state of odd-*A* nuclei (even number of a kind of nucleon and odd number of the other kind) is described by the spin and parity of that single odd nucleon
- Remark: Prediction is correct if we recognize that single hole in subshell gives the same *J* and π as single nucleon in same subshell



Quantum numbers: Ground state (3)

- For odd-proton/odd-neutron nucleus → rules of Brennan and Bernstein (based on the shell model) →

- Rule 1: when $j_1 = l_1 \pm ½$ and $j_2 = l_2 \mp ½$ → $J = |j_1 - j_2|$
- Rule 2: when $j_1 = l_1 \pm ½$ and $j_2 = l_2 \pm ½$ → $J = |j_1 \pm j_2|$
- Rule 3: states that for configurations in which the odd nucleons are a combination of particles and holes → $J = j_1 + j_2 - 1$

- Parity is given by $\pi = -1^{(l_1 + l_2)}$

Quantum numbers: Ground state (4)

Examples of application of the rules of Brennan and Bernstein:

- ^{38}Cl: 17 protons and 21 neutrons → the last proton is a $d_{3/2}$ level and the last neutron in a $f_{7/2}$ level →

  $j_p = 2 - ½$ / $j_n = 3 + ½$ → $J = |7/2 – 3/2| = 2$ / $\pi = -$
- ^{26}Al: 13 protons and 13 neutrons → the last proton and neutron are in $d_{5/2}$ hole states →

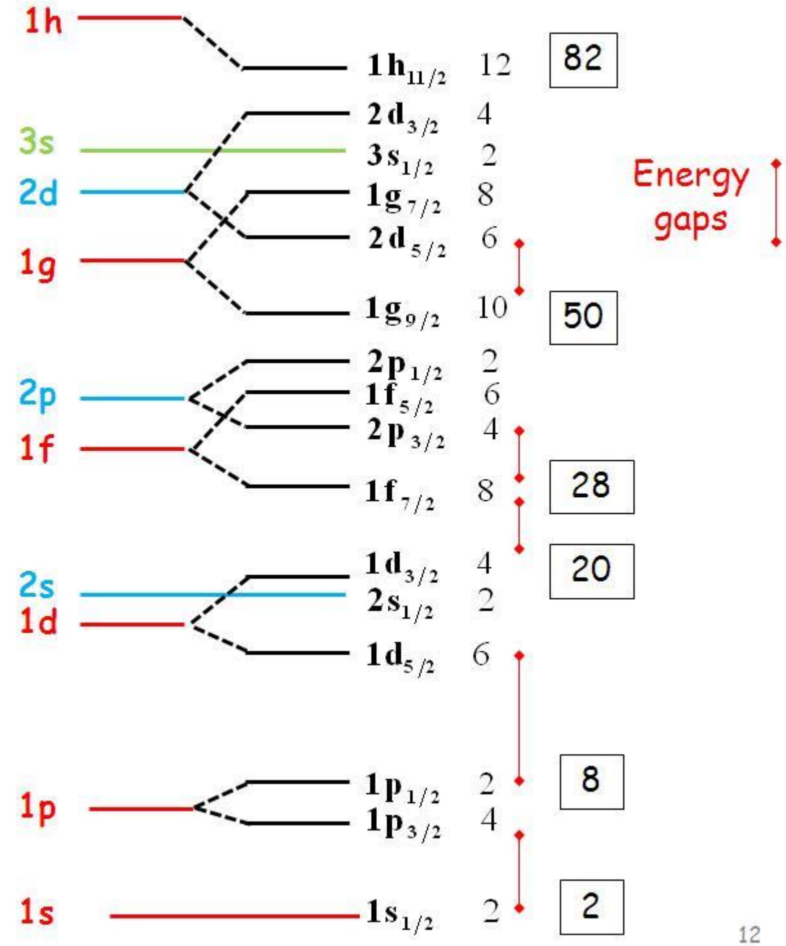
  $j_p = j_n = 2 + ½$ → $J = |5/2 + 5/2| = 5$ / $\pi = +$
- ^{56}Co: 27 protons and 29 neutrons → the last proton is in a $f_{7/2}$ hole state and the last neutron is in a $p_{3/2}$ state →

  $J = 7/2 + 3/2 - 1 = 4$ / $\pi = +$

Energy level pattern for nucleons

Ordering the nuclear orbitals

State Notation : $n\ell_j$



For shell model → nucleon levels are characterized by 3 numbers:

- n: the principal number
- l: the orbital angular momentum quantum number
- j: the total angular momentum quantum number such as j = l ± ½

Approximated good quantum numbers (1)

- Strong nuclear interaction → charge independence → particles affected equally by the strong force but with different charges (protons and neutrons) can be treated as different states of the same particle: the nucleon with a particular quantum number: the **isospin** (isotopic/isobaric spin) → value related to the number of charge states
- For a nucleon: 2 states → isospin quantum number t = ½ → 2 projections of the isospin → proton (p) has m_t = -½ and a neutron (n) has m_t = +½ (these projections are measured with respect to an arbitrary axis called the « 3-axis » in a system 1,2,3) → $t_3 = m_t\hbar$

Approximated good quantum numbers (2)

- Interpretation of isospin → the operator

$$q = e(\frac{1}{2} - t_3)$$

gives the charge $e(1/2 - m_t)$ of the nucleon

- Definition of raising and lowering operators →

$$\begin{aligned} t_+ &= t_1 + it_2 \\ t_- &= t_1 - it_2 \end{aligned} \quad \Longrightarrow \quad \begin{aligned} t_+ |p\rangle &= |n\rangle \\ t_+ |n\rangle &= 0 \\ t_- |n\rangle &= |p\rangle \\ t_- |p\rangle &= 0 \end{aligned}$$

Approximated good quantum numbers (3)

- We define the total isospin *T* of a nucleons system as →

$$T = \sum_{j=1}^{A} t_j$$

- All properties of angular momentum can be applied to *T* →

$$\begin{aligned} T^2 \left| T M_T \right\rangle &= T(T+1) \left| T M_T \right\rangle \\ T_3 \left| T M_T \right\rangle &= M_T \left| T M_T \right\rangle \end{aligned}$$

- The total charge operator Q of the system can be written →

$$Q = e(\frac{1}{2}A - T_3) \quad \Rightarrow \quad M_T = \frac{1}{2}A - Z = \frac{1}{2}(N - Z)$$

Approximated good quantum numbers (4)

- For a system of nucleons → isospin follows same rules than ordinary angular momentum vector → 2-nucleons system has total isospin $T = 0$ or 1 (corresponding to antiparallel or parallel orientations of the 2 isospins) → the 3-axis component of the total isospin vector T_3 is the sum of the 3-axis components of the individual nucleons
- For any nucleus →

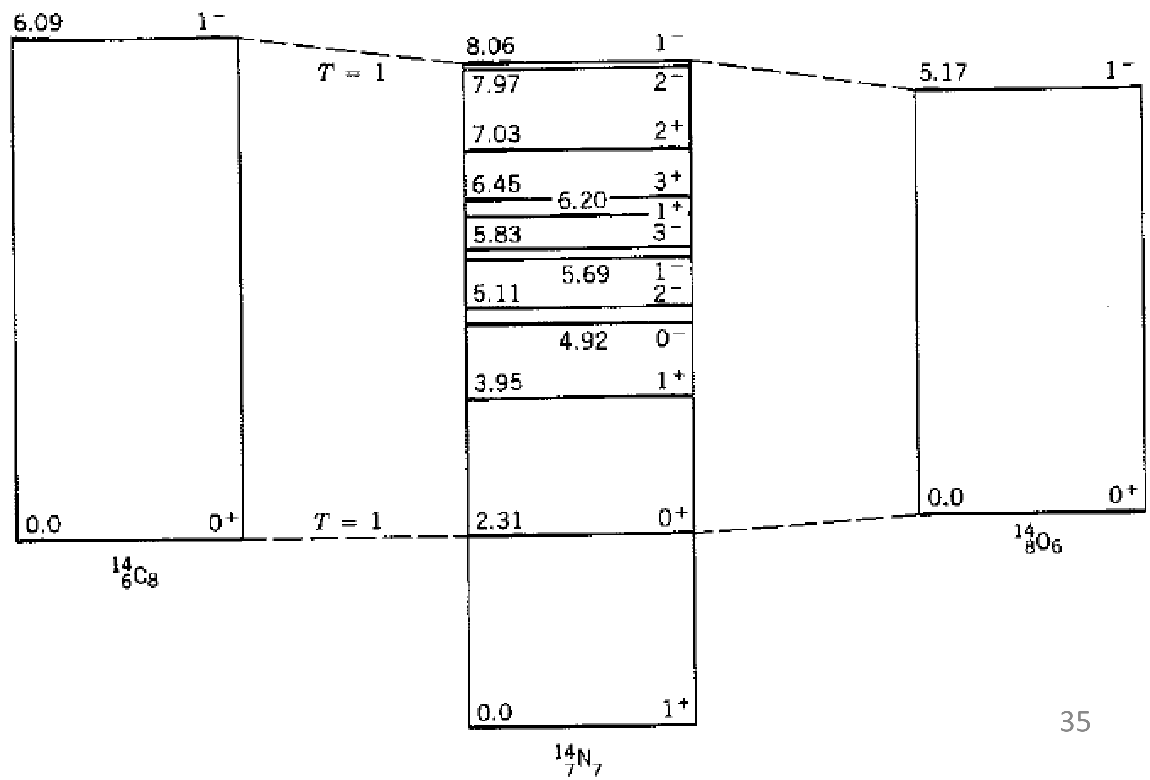
$$T_3 = \frac{1}{2}(N - Z) \leftrightarrow T \geq \frac{1}{2}|N - Z|$$

- Example: 2-nucleons system → p-p: $T_3 = -1$ ($T = 1$), n-n: $T_3 = +1$ ($T = 1$), p-n: $T_3 = 0$ ($T = 0$ or $T = 1$)

Approximated good quantum numbers (5)

- If perfect charge independence (and electromagnetic interaction is not considered) → the isospin quantum number *T* gives the number *2T + 1* of isobars with this particular level in their spectrum with same quantum numbers *J* and π → notation:

$$J^\pi; T \leftrightarrow J^\pi; T, E_x$$



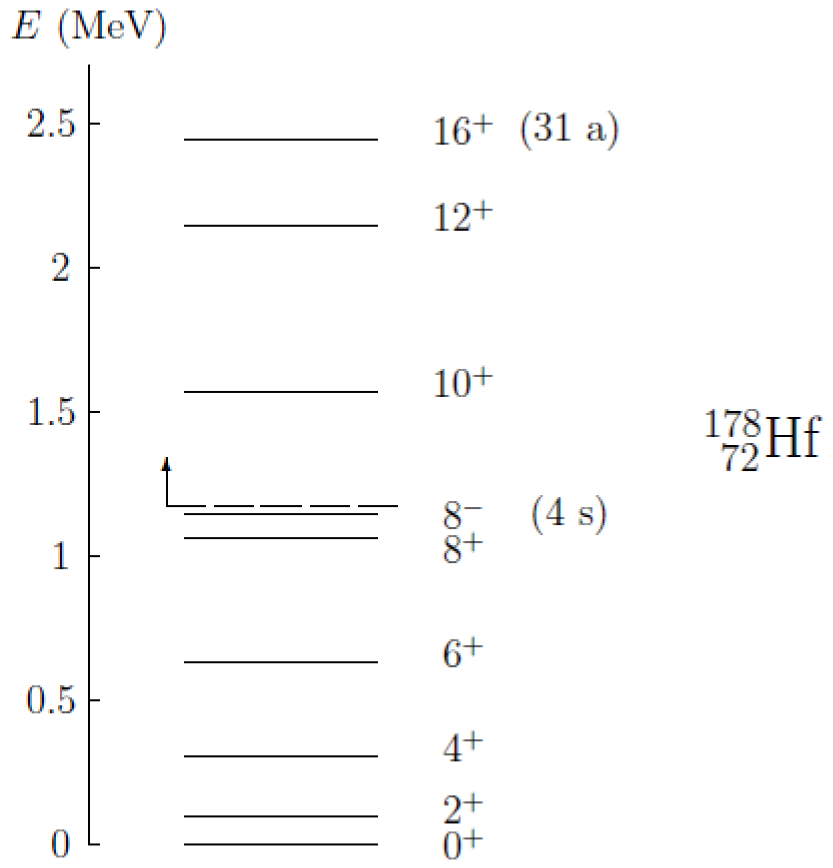
Energy levels of ^{14}C and ^{14}O are shifted by 2.36 and 2.44 MeV ↔ ≠ $m_{p,n}$ + Coulomb → ^{14}C and ^{14}O have *T* = 1 while ^{14}N has *T* = 0 except *T* = 1 for levels at 2.31 and 8.06 MeV

Spectrum: Nuclear levels (1)

- ≠ types of energy levels
- Some are bound states → no spontaneous dissociation → de-excitation to levels with smaller energy by emitting radiation
- Some are resonances → there are beyond the dissociation threshold → dissociation or de-excitation
- Lifetimes of nuclear excited states are typically in the range $10^{-15} - 10^{-14}$ s → with few exceptions only nuclei in the ground state are present on Earth.
- The rare excited states with large lifetimes (> 1 s) are called isomeric states (or isomers or metastable states) → isomeric state of nucleus ^{A}X is designed by ^{Am}X
- Isomer has generally J and π very different from states with smaller energy

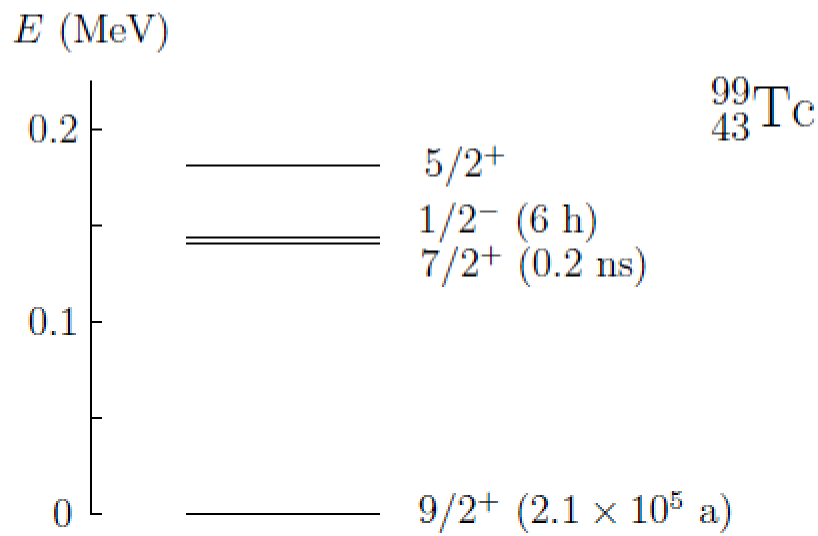
Spectrum: Nuclear levels (2)

- Ground state 0^+ = stable
- Level 8^- (1.147 MeV): τ = 4 s
- Level 16^+ (2.446 MeV): τ = 31 years (de-excitation to 12^+ → ≠ of $4\hbar$ →small probability)



Spectrum: Nuclear levels (3)

- Ground state $9/2^+$ = unstable but long
- Level $1/2^- \rightarrow 7/2^+$
- Level $7/2^+$ (0.141 MeV): quick de-excitation by γ emission to $9/2^+ \rightarrow$ application in nuclear medicine



- Extreme example → the first exited state of ^{180}Ta has a lifetime τ = 10^{15} years while the ground state β-decays with τ = 8 hours → All ^{180}Ta present on Earth is therefore in the excited state

Nuclear radius (1)

- Previously → definition of the charge radius →

$$\langle r^2 \rangle_{ch}^{1/2} = \sqrt{\int r^2 \rho_{ch}(\boldsymbol{r}) d\boldsymbol{r}}$$

- The charge density of a nucleon is measured from the analysis of high energy electrons elastically scattered from it ↔ distance ≈ 0.1 fm → reduced de Broglie wave length $\lambda/2\pi = \hbar/p \approx 0.1$ fm → $E \approx pc \approx 2000$ MeV
- Initial electron wave function: $\exp(i\boldsymbol{k}_i\boldsymbol{r})$ (free particle of momentum $\boldsymbol{p}_i = \hbar\boldsymbol{k}_i$); scattered electron (also free particle with momentum $\boldsymbol{p}_f = \hbar\boldsymbol{k}_f$): $\exp(i\boldsymbol{k}_f\boldsymbol{r})$
- As elastic collision → $|\ \boldsymbol{p}_i\ | = |\ \boldsymbol{p}_f\ |$

Nuclear radius (2)

- According to the Fermi Golden Rule → probability of transition is $\propto$ to the square of *F* (with *F(0)* = 1)→

$$F(\boldsymbol{k}_i, \boldsymbol{k}_f) = \int \psi_f^* V(r) \psi_i d\boldsymbol{r} \leftrightarrow F(\boldsymbol{q}) = \int e^{i\boldsymbol{qr}} V(r) d\boldsymbol{r}$$

- $\boldsymbol{q} = \boldsymbol{k}_i - \boldsymbol{k}_f$ is the momentum change of scattered electron
- *V(r)* depends on the nuclear charge density $Ze\rho_{ch}(\boldsymbol{r}')$ →

$$V(r) = -\frac{Ze^2}{4\pi\epsilon_0} \int \frac{\rho_{ch}(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} d\boldsymbol{r}'$$

- With $\boldsymbol{qr} = qr\sin\theta$ and integrating on $\boldsymbol{r}$ → normalized $F(\boldsymbol{q})$ (called *form factor*) is:

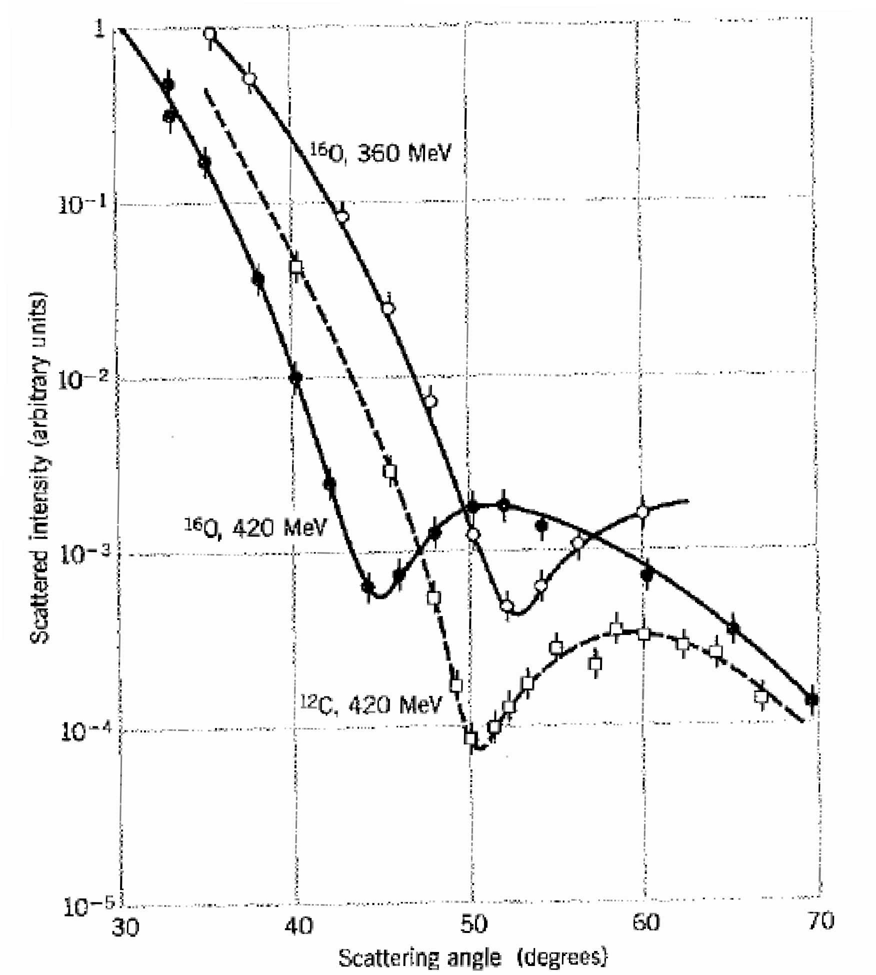
$$F(\boldsymbol{q}) = \int e^{i\boldsymbol{qr}} \rho_{ch}(\boldsymbol{r}') d\boldsymbol{r}'$$

Nuclear radius (3)

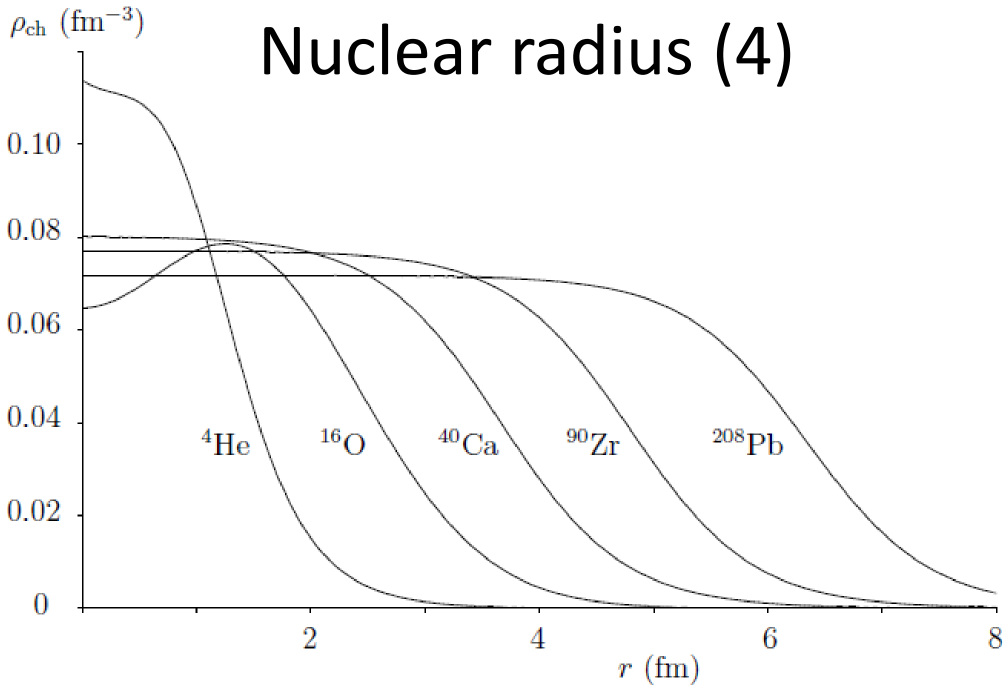
- If $\rho_{ch}(\boldsymbol{r'})$ depends only on r' (not on $\theta'$and ϕ') →

$$F(q) = \frac{4\pi}{q} \int \sin(qr')\rho_{ch}(r')r'dr'$$

- As $|\,\boldsymbol{p}_i\,| = |\,\boldsymbol{p}_f\,|$ → q = f(α) with α the angle between $\boldsymbol{p}_i$ and $\boldsymbol{p}_f$ → $q = (2p/\hbar) \sin \alpha/2$ → the measure of α → ρ_{ch}



Nuclear radius (4)



- Result for various nuclei → the central nuclear charge density is nearly the same for all nuclei → nucleons do not congregate at the center → nucleons are piled up as spheres ↔ short range of nuclear force
- The number of nucleons by unit volume is roughly constant → with *R* the mean nuclear radius of a sphere of uniform density with the same charge radius as the nucleus →

$$\frac{A}{4/3\pi R^3} \sim \text{ constant} \leftrightarrow R = R_0 A^{1/3}$$

Nuclear radius (5)

- We can also define the matter radius *r* that is the root mean square radius of the distribution of nucleons such as →

$$\langle r^2 \rangle^{1/2} = \sqrt{\int r^2 \rho(\boldsymbol{r}) d\boldsymbol{r}}$$

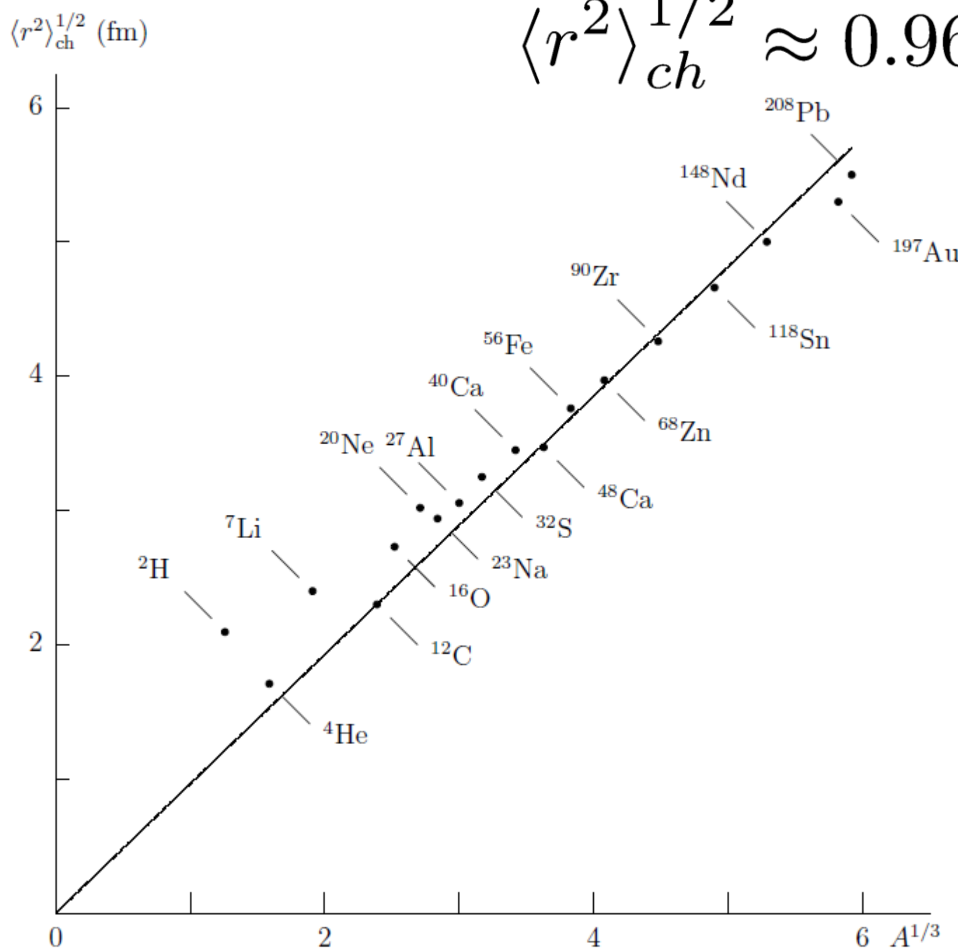
- For a sphere of constant density

$$\langle r^2 \rangle = \frac{\int r^2 \rho(\boldsymbol{r}) d\boldsymbol{r}}{\int \rho(\boldsymbol{r}) d\boldsymbol{r}} = \frac{\int_0^R r^4 dr}{\int_0^R r^2 dr} = \frac{3}{5} R^2$$

Nuclear radius (6)

- From experimental measurements of ρ_{ch} and considering that the charge radius also follows a law in $A^{1/3}$ →

$$\langle r^2 \rangle_{ch}^{1/2} \approx 0.96 A^{1/3}$$



Assuming that $r_{ch} \approx r$ (true for no-exotic nuclei) → R_0 = 1.24 fm

Nuclear electromagnetic moments (1)

- Previous expressions are obtained for a sphere of constant density
- More precise calculations imply to consider a density obtained from the Wigner-Eckart theorem:

$$\rho(\boldsymbol{r}) = \sum_{\lambda \text{ pair}=0}^{2J} P_\lambda(\cos\theta_r)\rho^{(\lambda)}(r)$$

$$\rho^{(\lambda)}(r) = \frac{2\lambda+1}{4\pi r^2} \langle \Psi^{JJ\pi} | \sum_{\rho=1}^{Z} \delta(r'-r) P_\lambda(\cos\theta') | \Psi^{JJ\pi} \rangle$$

- The charge density is pair and has a rotational symmetry about *z*-axis
- For $J = 0 \rightarrow \rho(\boldsymbol{r}) = \rho^{(0)}(r)$

Nuclear electromagnetic moments (2)

- For $J \neq 0 \rightarrow$ there is a measurable quantity which gives the difference between a spherical charge distribution and the real charge distribution $\rightarrow$ the electric-quadrupole moment
- In a general way the electric-multipole moment is written:

$$Q^{(\lambda)} = 2e \int r^{\lambda} P_{\lambda}(\cos\theta_r)\rho(\boldsymbol{r})d\boldsymbol{r} = \frac{8\pi e}{2\lambda+1}\int_0^{\infty} r^{\lambda+2}\rho^{(\lambda)}(r)dr$$

- Electric-multipole moments are the moments of the charge density $\rightarrow$ as charge density is pair $\rightarrow$ only pair moments exist
- For $\lambda = 0 \rightarrow$ we obtain the trivial value $Q^{(0)} = 2Ze \propto$ to the total charge

Nuclear electromagnetic moments (3)

- Due to the orthogonality of Legendre polynomials → multipole moment is ≠ 0 only if $J \geq \lambda/2$
- Moreover parity conservation implies → even λ
- Only even multipole moments lower or equal to $2J$ give a non-zero value of $Q^{(\lambda)}$ →in particular electric-dipole moment of a nucleus is zero (considering that the parity is a good quantum number)
- The first non-trivial moment is the electric-quadrupole moment (λ = 2)

Electric-quadrupole moment

- The electric-quadrupole moment can be written:

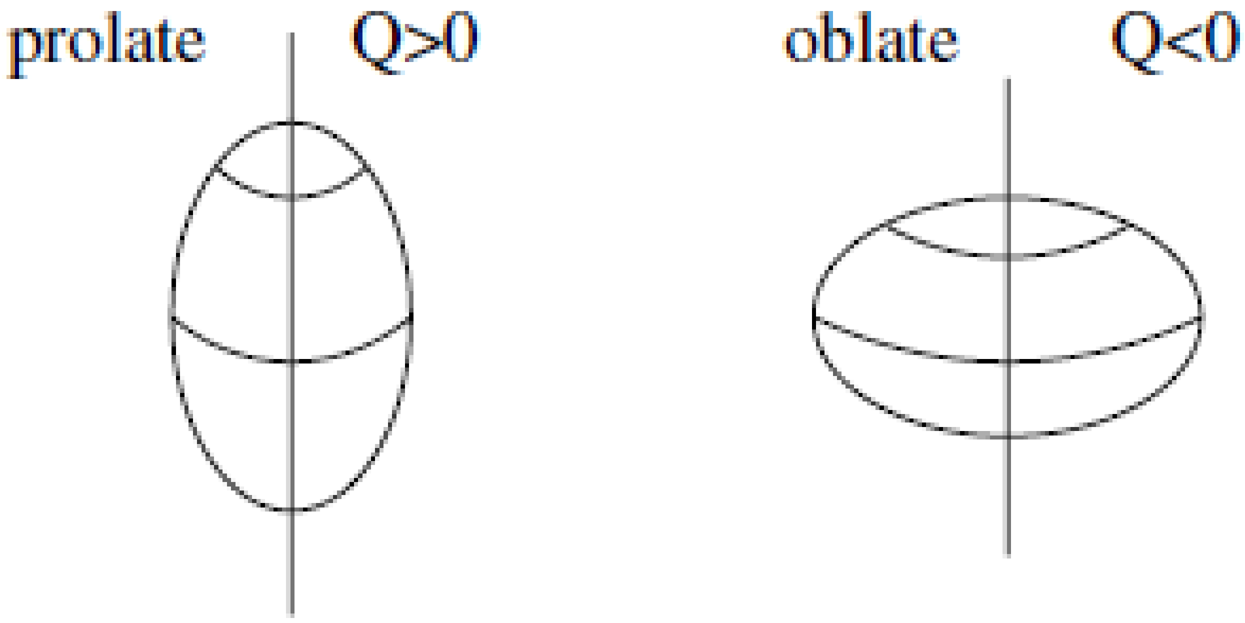
$$Q^{(2)} = 2e \int r^2 P_2(\cos\theta_r)\rho(\boldsymbol{r})d\boldsymbol{r}$$

- We can write

$$r^2 P_2(\cos\theta) = \frac{1}{2}(3z^2 - r^2) = \frac{1}{2}(2z^2 - x^2 - y^2)$$

- For $Q^{(2)} > 0$ → nucleus is deformed in the direction of z → rugby ball shape (prolate)
- For $Q^{(2)} < 0$ → nucleus is deformed in the plane $\perp$ to z → cushion shape (oblate)
- For $Q^{(2)} = 0$ → no deformation of the nucleus → spherical shape

Nucleus deformation



Magnetic-dipole moment (1)

- Magnetic-multipole moments are other moments characteristic of magnetic properties of the nucleus
- They come from the magnetization density
- The most important is the magnetic-dipole moment (simply called magnetic moment)
- The operator magnetic-dipole moment is

$$\boldsymbol{M} = \sum_{i=1}^{A} (g_{li} \boldsymbol{L}_i' + g_{si} \boldsymbol{S}_i)$$

with g_{li} = 1 for a proton and 0 for a neutron, and g_{si} are the gyromagnetic ratios (g_{sp} = 5.5856947 and g_{sn} = -3.826085)

Magnetic-dipole moment (2)

- This operator is a combination of the operators L'_i (orbital kinetic moment) and S_i (spin) of each nucleon
- The magnetic-dipole moment is defined by (with μ_N, the Bohr magneton):

$$\mu = \frac{\mu_N}{\hbar} \left\langle \Psi^{JJ\pi} \right| M_z \left| \Psi^{JJ\pi} \right\rangle$$

- From the Wigner-Eckart theorem → the nucleus has a magnetic moment if $J \geq 1/2$

Atomic mass

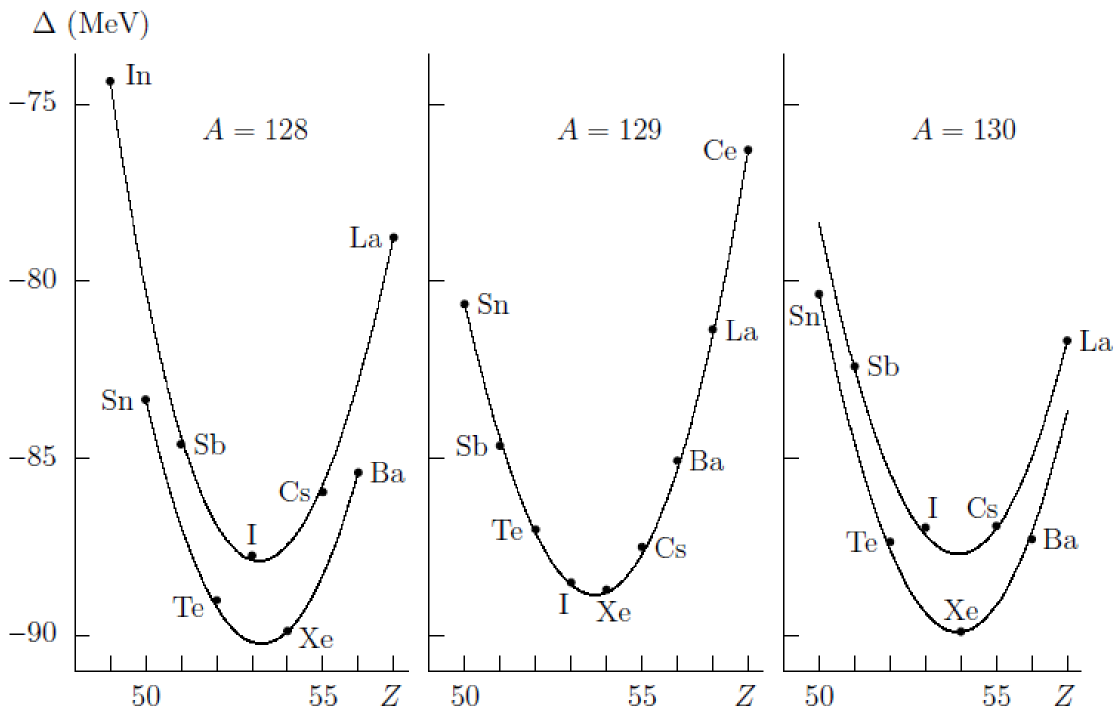
- Mass is bound to energy conservation → important to define stability of nuclei
- We define the *atomic mass*: mass of a neutral atom in ground state → *M(A,Z)* or $M(^{A}X)$
- 1 unified *atomic* mass unit (u) = $1.6605390 \times 10^{-27}$ kg = 931.4940 MeV/c^2 = 1/12 of M(12,6) (*atom* of ^{12}C)
- Example:
  - $M(^{1}H)$ = 1.007825032 u
  - m_p = 1.007276467 u
  - m_e = 5.48579909 10^{-4} u
  - $M(^{1}H) = m_p + m_e$ - 1 Rydberg (electron binding energy must be considered)

Mass excess

- Mass atomic is often given in *mass excess* form: *Δ(A,Z)* (energy expressed in MeV) →

$$\Delta(A,Z) = [M(A,Z) - A]uc^2$$

- *Δ* for isobar families (fixed A) as a function of Z varies only a little →



Parabolic in shape for odd-*A* and double parabola for even-*A*

Nuclear mass

- The *nuclear mass* is the mass of the nucleus of an isotope in ground state → $m(A,Z)$ or $m(^{A}X)$
- We have →

$$m(A,Z) = M(A,Z) - Zm_e + B_e(Z)/c^2$$

- The electron binding energy B_e decreases the total mass of the atom
- $B_e = \sum_i B_i$ with $B_i = a_i(Z-c_i)^2$ (a_i and c_i are constant parameters for each electron shell
- B_e is generally neglected in the definition of nuclear mass (in first approximation) because it is quite smaller than usual nuclear energies ($B_e \sim$ 10-100 keV ↔ $M(A,Z) \sim A \times 1000$ MeV)

Binding energy of a nucleus (1)

- The binding energy *B* of a nucleus is defined as the negative of the difference between the nuclear mass and the sum of the masses of the constituents →

$$B(A,Z) = [Nm_n + Zm_p - m(A,Z)]c^2$$

- *B* is positive for all nuclei (stable or unstable) → implies that the nucleus does not spontaneously break down into its all constituents (but does not imply that it is stable)
- We can write (^{1}H is the hydrogen atom and $B_e(1)$ = 13.6 eV) →

$$\begin{aligned} B(A,Z) &= [Nm_n + ZM(^1H) - m(A,Z)]c^2 - B_e(Z)Z + B_e(1) \\ &\simeq [Nm_n + ZM(^1H) - m(A,Z)]c^2 \end{aligned}$$

- By adding and subtracting *A* = *N* + *Z* →

$$B(A,Z) = N\Delta(^1n) + Z\Delta(^1H) - \Delta(A,Z)$$

Separation energy

- Analogous to ionization energy in atomic physics → definition of the neutron/proton separation energy = amount of energy that it is needed to remove a neutron/proton from a nucleus ${}^{A}_{Z}X_N$

$$\begin{aligned} S_n &= B({}^{A}_{Z}X_N) - B({}^{A-1}_{Z}X_{N-1}) \\ &\simeq [m({}^{A-1}_{Z}X_{N-1}) - m({}^{A}_{Z}X_N) + m_n]c^2 \end{aligned}$$

Table 3.1 Some Mass Defects and Separation Energies

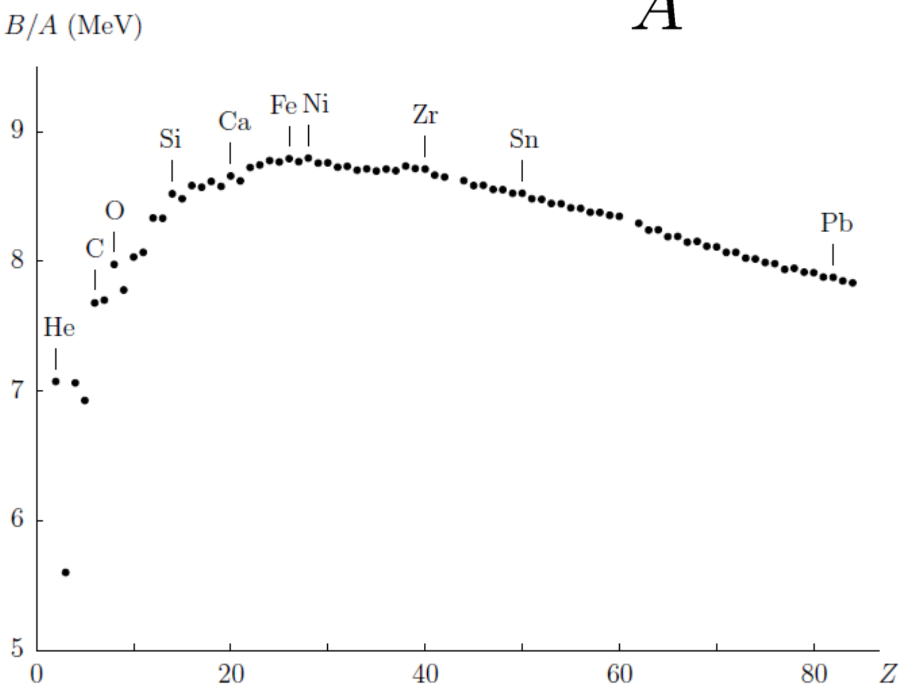
Nuclide	Δ (MeV)	S_n (MeV)	S_p (MeV)
^{16}O	−4.737	15.66	12.13
^{17}O	−0.810	4.14	13.78
^{17}F	+1.952	16.81	0.60
^{40}Ca	−34.847	15.64	8.33
^{41}Ca	−35.138	8.36	8.89
^{41}Sc	−28.644	16.19	1.09
^{208}Pb	−21.759	7.37	8.01
^{209}Pb	−17.624	3.94	8.15
^{209}Bi	−18.268	7.46	3.80

$$\begin{aligned} S_p &= B({}^{A}_{Z}X_N) - B({}^{A-1}_{Z-1}X_{N}) \\ &\simeq [m({}^{A-1}_{Z-1}X_{N}) - m({}^{A}_{Z}X_N) + m({}^{1}H)]c^2 \end{aligned}$$

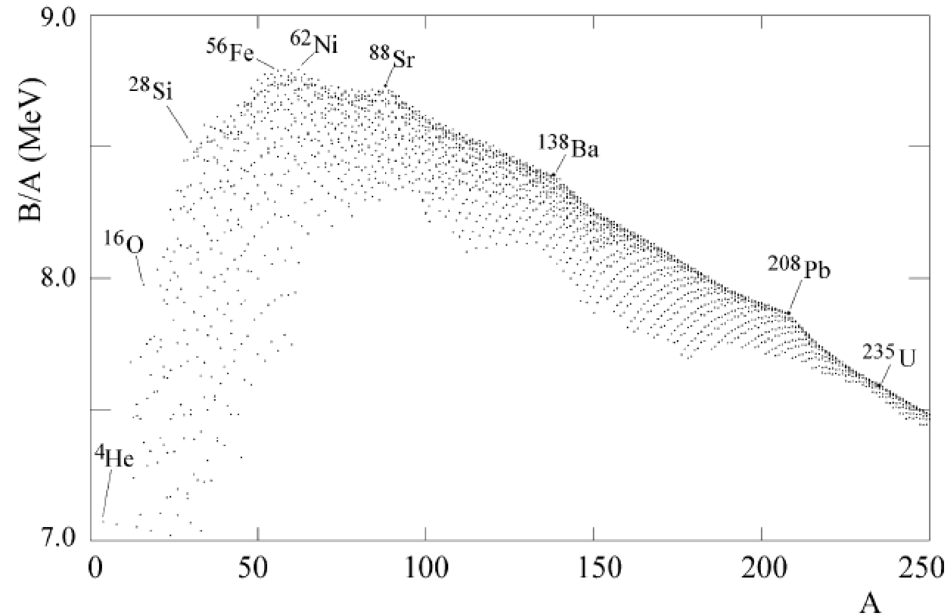
Binding energy of a nucleus (2)

- As first approximation for stable nuclei (with $A \gtrsim 12$) →

$$\frac{B(A,Z)}{A} \approx (8.3 \pm 0.5)\ \mathrm{MeV}$$



Stable nuclei



Stable and unstable nuclei

Binding energy of a nucleus (3)

- This property is explained by the short range of nuclear force
- Indeed for a long range force (as Coulomb force) → the binding energy of a *n*-particles system is $\propto$ to the number of particles pairs → $B(n) = (1/2)n(n-1)$
- As the binding energy of a nucleus as not this trend → nuclear interaction has the *saturation* property → each nucleon may only interact with a limited number of close nucleons
- The binding energy by nucleon is fixed by the numbers of neighbours → independent on the size of the nucleus
- If the nuclei is too small → saturation is not reached

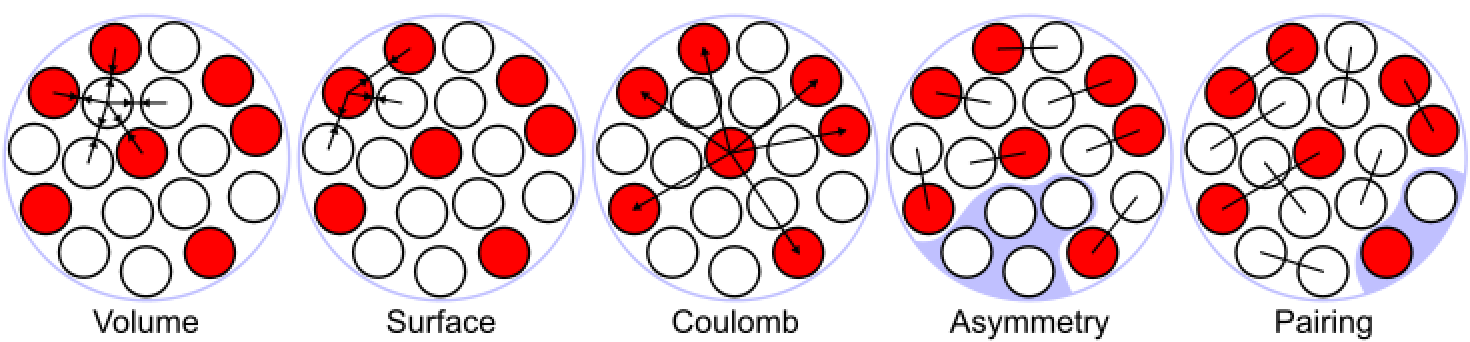
Binding energy of a nucleus: Bethe-Weizsäcker formula

- Semi-empirical expression based on simplified physical arguments and on a fitting to data → valid for absolutely stable nuclei
- Physical model beyond it → liquid drop model → the nucleus is treated as a drop of incompressible nuclear fluid of very high density

$$B(A,Z) = a_V A - a_S A^{2/3} - a_C \frac{Z(Z-1)}{A^{1/3}} - a_a \frac{(N-Z)^2}{A} + \delta$$

$a_V \simeq 15.56$ MeV $\quad$ $a_C \simeq 0.72$ MeV

$a_S \simeq 17.23$ MeV $\quad$ $a_a \simeq 23.285$ MeV



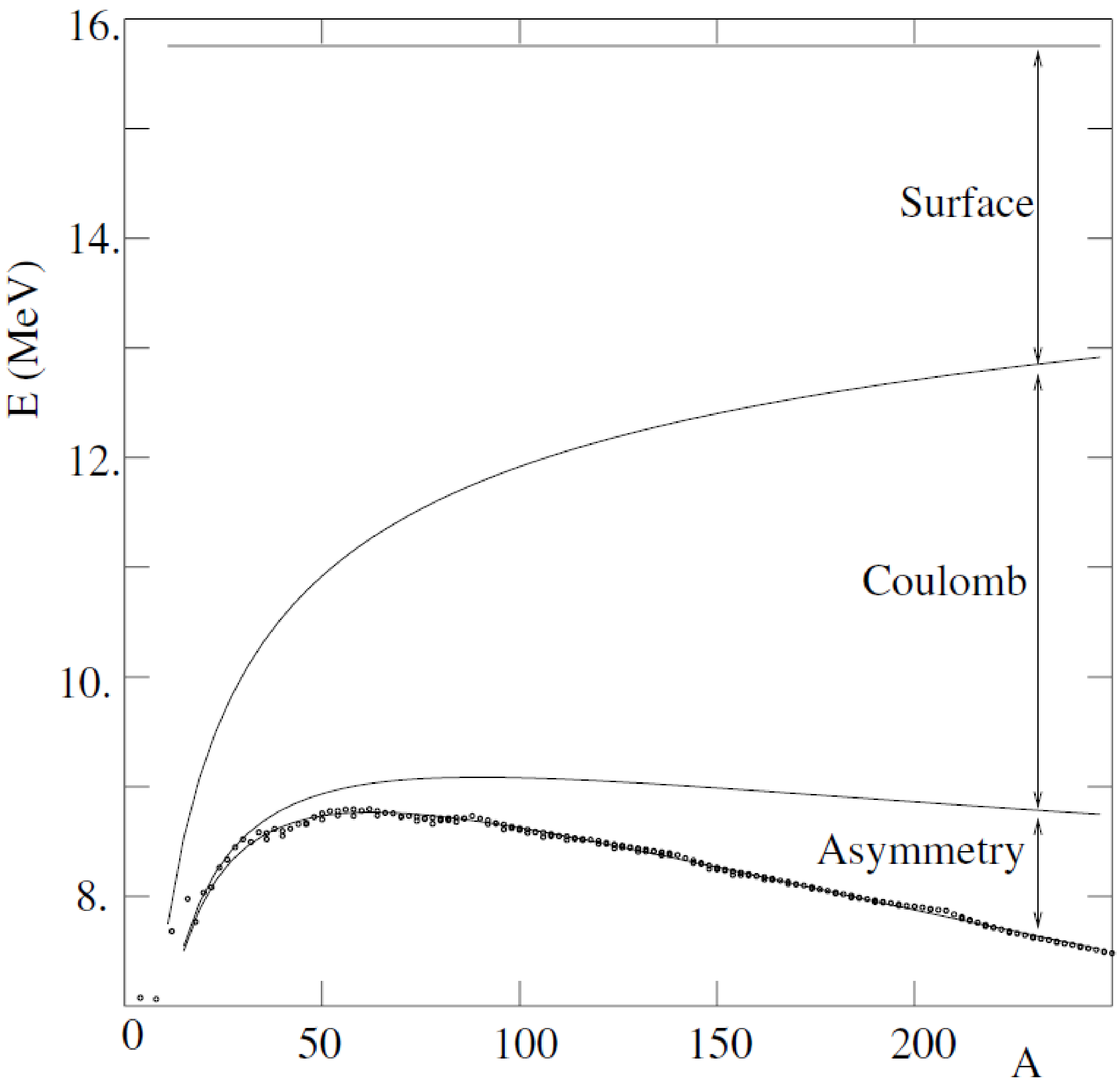
Bethe-Weizsäcker formula (1)

- $a_V A$: Volume term → reflects the saturation property → each nucleon interacts only with nearest-neighbours → constant binding energy per nucleon *B/A*
- $-a_S A^{2/3}$: Surface term → lowers the binding energy → nucleons near the surface feel forces coming only from the inside of the nucleus → their contribution in first term is overestimated → $\propto$ to the area $4\pi R^2 \sim A^{2/3}$
- $-a_C Z(Z-1) A^{-1/3}$: Coulomb repulsion term → long range force due to protons → Coulomb energy E_C of a sphere of charge *Ze* and radius *R* → $E_C = \frac{3}{5}\frac{Z^2 e^2}{4\pi\epsilon_0 R}$ → as this energy is $\propto$ to number of protons pairs → Z^2 must be replaced by $Z(Z-1)$ → $a_C = 0.6e^2/4\pi\epsilon_0 R \approx 0.72$ MeV (with $R = 1.24$ fm) → it favors a neutron excess over protons
- $-a_N (N-Z)^2 A^{-1}$: Asymmetry term → due to Pauli principle (isospin) the minimum energy in a nucleus is reached for $N \approx Z$ (otherwise we could have $Z = 2$ and $N = 100$) → if proton was not charged we would exactly $N = Z$ but due Coulomb repulsion $N \geq Z$ → for small *A* → $N = Z$ and for large *A* → $N > Z$ → asymmetry term $\propto$ to the difference between *N* and Z → Fermi model gives $(N-Z)^2/A$

Bethe-Weizsäcker formula (2)

- δ: Pairing term → as seen before → nucleons have tendency to couple pairwise to establish stable configuration →
  - Odd *A*: δ = 0 by definition → less favorable than N-even and Z-even but more favorable than *N*-odd and *Z*-odd
  - *Z*-even/*N*-even: all nucleons may be paired → the bonding is favored (δ > 0) → empirical expression $\delta = +12A^{-1/2}$ MeV
  - *Z*-odd/*N*-odd: one neutron and one proton cannot be paired → the binding energy is decreased (δ < 0) → empirical expression $\delta = -12A^{-1/2}$ MeV
- This pairing term has important consequences on the stability of nuclei → Among the 275 stable known nuclei → 166 are *Z*-even/*N*-even – 55 are *Z*-even/*N*-odd – 50 are *Z*-odd/*N*-even – 4 are *Z*-odd/*N*-odd: ^{2}H, ^{6}Li, ^{10}B, ^{14}N (attention a lot of unstable Z-odd/N-odd nuclei exist)

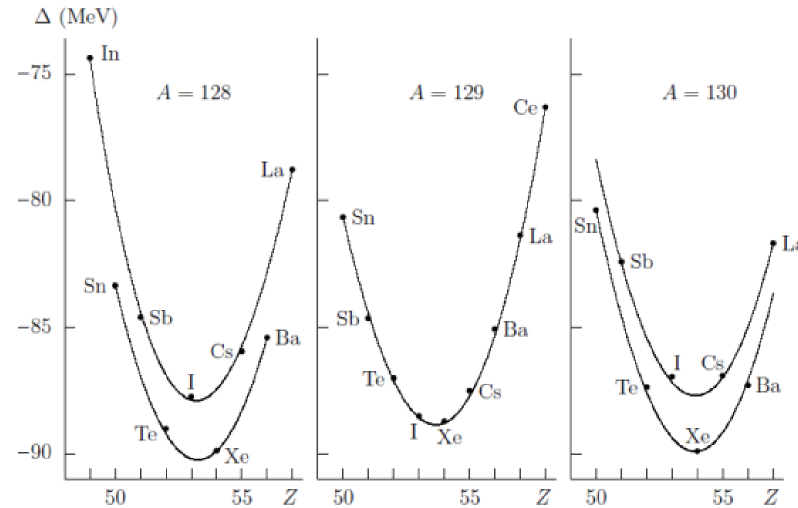
Bethe-Weizsäcker formula (3)



- The observed binding energies as a function of *A* and the predictions of the mass formula
- Only even–odd combinations of *N* and *Z* are considered → pairing term vanishes

Bethe-Weizsäcker formula (4)

- The Bethe-Weizsäcker formula explains the parabolic behaviour for the masses → for A = constant → second order polynomial in Z → stability valley



- The parabola is centered about the point where the equation

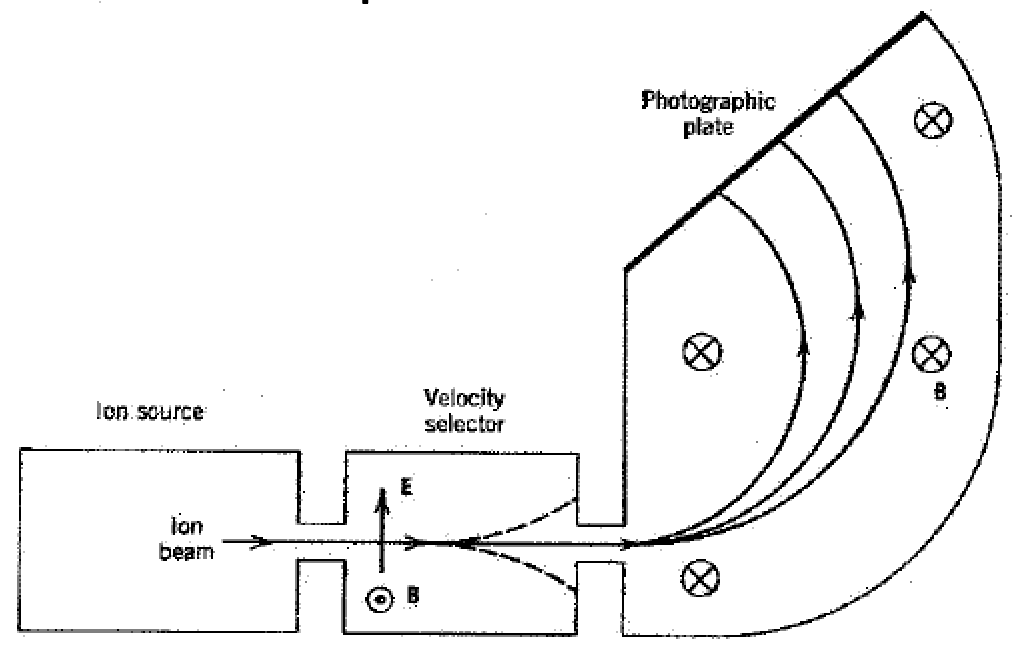
$$m(A,Z) \simeq N m_n + Z M(^1H) - B(A,Z)/c^2$$

reaches the minimum ∂M/∂Z = 0 →

$$Z_{min} = \frac{[m_N - m(^1H)] + a_C A^{-1/3} + 4a_a}{2a_C A^{-1/3} + 8a_a A^{-1}} \simeq \frac{A}{2}\frac{1}{1+0.0078A^{2/3}}$$

- The splitting for even-A is due to pairing acting in opposite directions for even-even nuclei (lower parabola) and odd-odd nuclei (upper parabola)

Mass spectrometer



- Production of an ion beam with thermal distribution of velocities
- A selector passes only ions with a particular velocity v
- Momentum selection by magnetic field B permits mass identification → $r = mv/qB$

Nuclei stability: Stability in particles

- Different notions of stability
- First definition: **stable in particles** → no possible dissociation in sub-systems with smaller total energy
- We consider $(A,Z) \rightarrow (A_1,Z_1) + (A_2,Z_2)$ with $A = A_1 + A_2$ and $Z = Z_1 + Z_2$ → stable if ($\forall$ A_1, Z_1):

$$m(A,Z) < m(A_1,Z_1) + m(A_2,Z_2) \text{ or } B(A,Z) > B(A_1,Z_1) + B(A_2,Z_2)$$

- The nuclear mass can be replaced by the atomic mass except in some cases where the stability depends on the presence of the e^-
- This case corresponds to a spontaneous fission, to the emission of α, neutron, proton,… → **instability in particles** → *A* changes → lifetime generally very short $\sim 10^{-21}$ s

Nuclei stability: Absolute stability

- More generally → a nucleus is **absolutely stable** if ($\forall$ m_i):

$$m(A, Z) < \sum_i m_i$$

- The sum concerns all possible masses and all possible disintegration modes
- If absolute stability → stability in particles

Nuclei stability: Instability by β emission

- Attention → the previous definitions are not sufficient

- Other instabilities exist due to the weak force (β disintegration) → **instability by β emission**

- If β emission → A is constant but N and Z change

- Variable lifetime from 10^{-6} s to 10^{15} years

Nuclei stability: Examples (1)

- ^{12}C: no possible splitting emitting energy from ground state, no β emission → stable
- ^{8}Be: can split into 2 ^{4}He → $M(8,4) - 2M(4,2) \approx 0.092$ MeV → unstable (lifetime ≈ 10^{-16} s)
- ^{3}H: stable in particles but by β disintegration → ^{3}He (lifetime ≈ 12.3 years)
- Information may be deduced from →

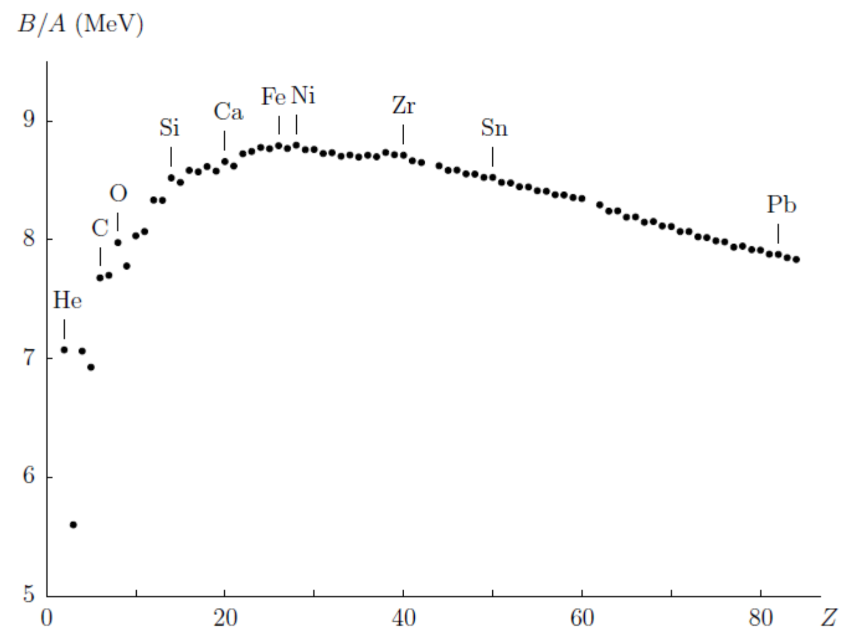
$$B(A,Z) \quad > \quad B(A_1,Z_1) + B(A_2,Z_2)$$

$$\Leftrightarrow$$

$$A_1\left(\frac{B(A,Z)}{A} - \frac{B(A_1,Z_1)}{A_1}\right) \quad + \quad A_2\left(\frac{B(A,Z)}{A} - \frac{B(A_2,Z_2)}{A_2}\right) > 0$$

Nuclei stability: Examples (2)

- This relation is fulfilled for nucleus corresponding to the maximum (Fe) of the curve and for nuclei at the left of the maximum ↔ their fragments have smaller *B/A* ratios → some nuclei are certainly stable
- For heavy nuclei → completely ≠ → they are beyond the maximum → they provide energy during splitting → the 2 terms of previous equation are negative



Nuclei stability: Examples (3)

- ^{208}Pb: various dissociations seems possible: 2 examples:

$$^{208}Pb \rightarrow \alpha + {}^{204}Hg + 0.52\ \mathrm{MeV}$$

$$^{208}Pb \rightarrow 2^{104}Zr + 110\ \mathrm{MeV}$$

- Practically → none of these processes is observed and ^{208}Pb is stable (only one heavier nucleus is stable: ^{209}Bi)
- The lifetime of ^{208}Pb is so long → disintegrations can be considered as negligible → stable
- One of the reasons for this long lifetime is that the system of nucleons has to cross the potential barrier to split up → the probability of crossing may be so small that the lifetime is extremely long

Nuclei stability: Conventional stability

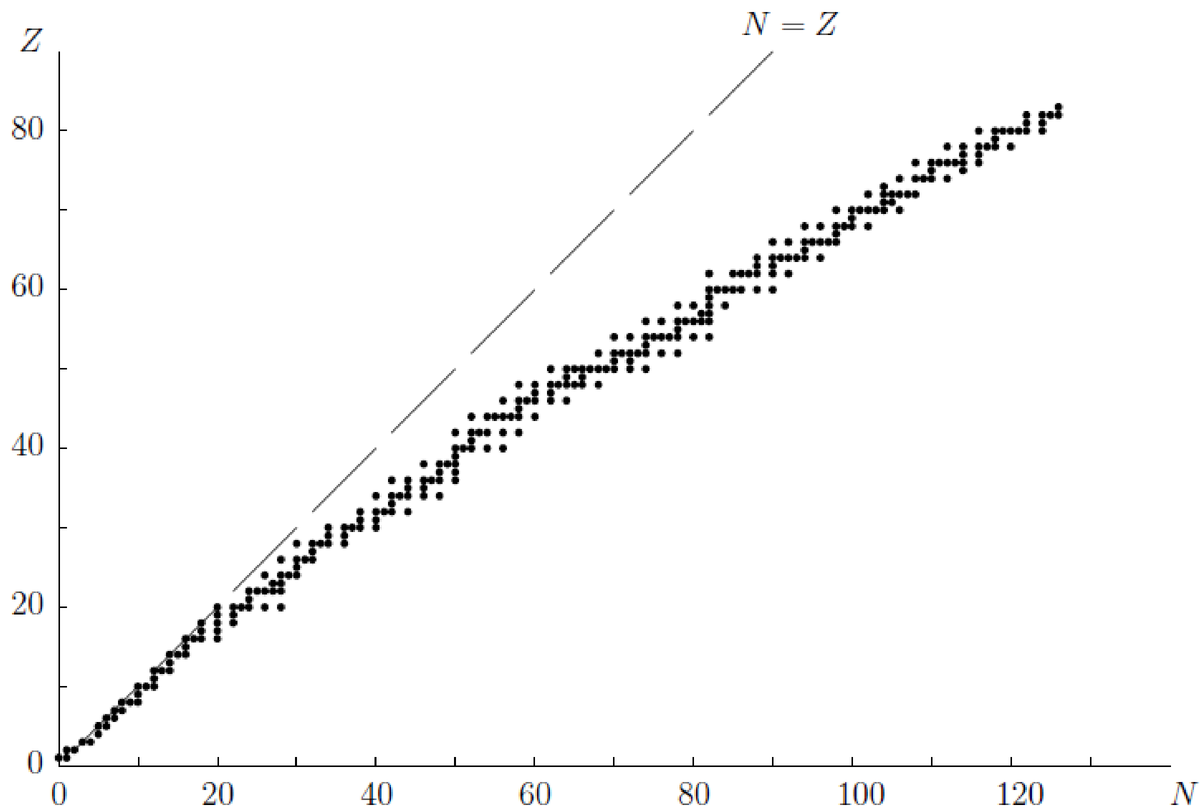
- Finally the convention is to consider that a nucleus is **stable if its lifetime is larger than the age of Universe** →

$$\tau \gg 1.5 \times 10^{10} \text{ years} \simeq 5 \times 10^{17} \text{ s}$$

- Practical definition even though it is artificial
- Example: ^{209}Bi has τ = 1.9 10^{19} years → stable but disintegration was observed:

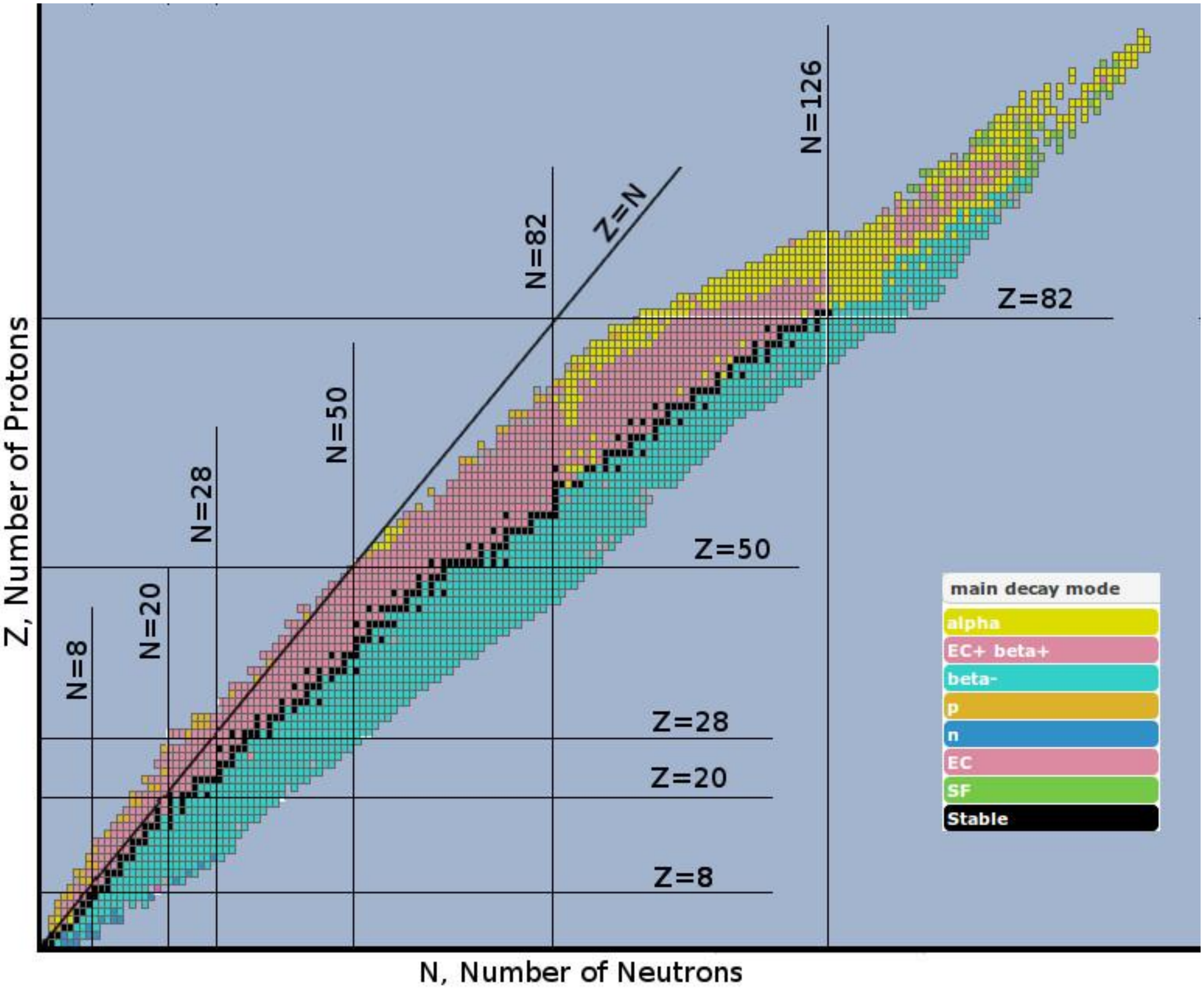
$$^{209}Bi \to \alpha + {}^{205}Tl$$

Stable nuclei (1)

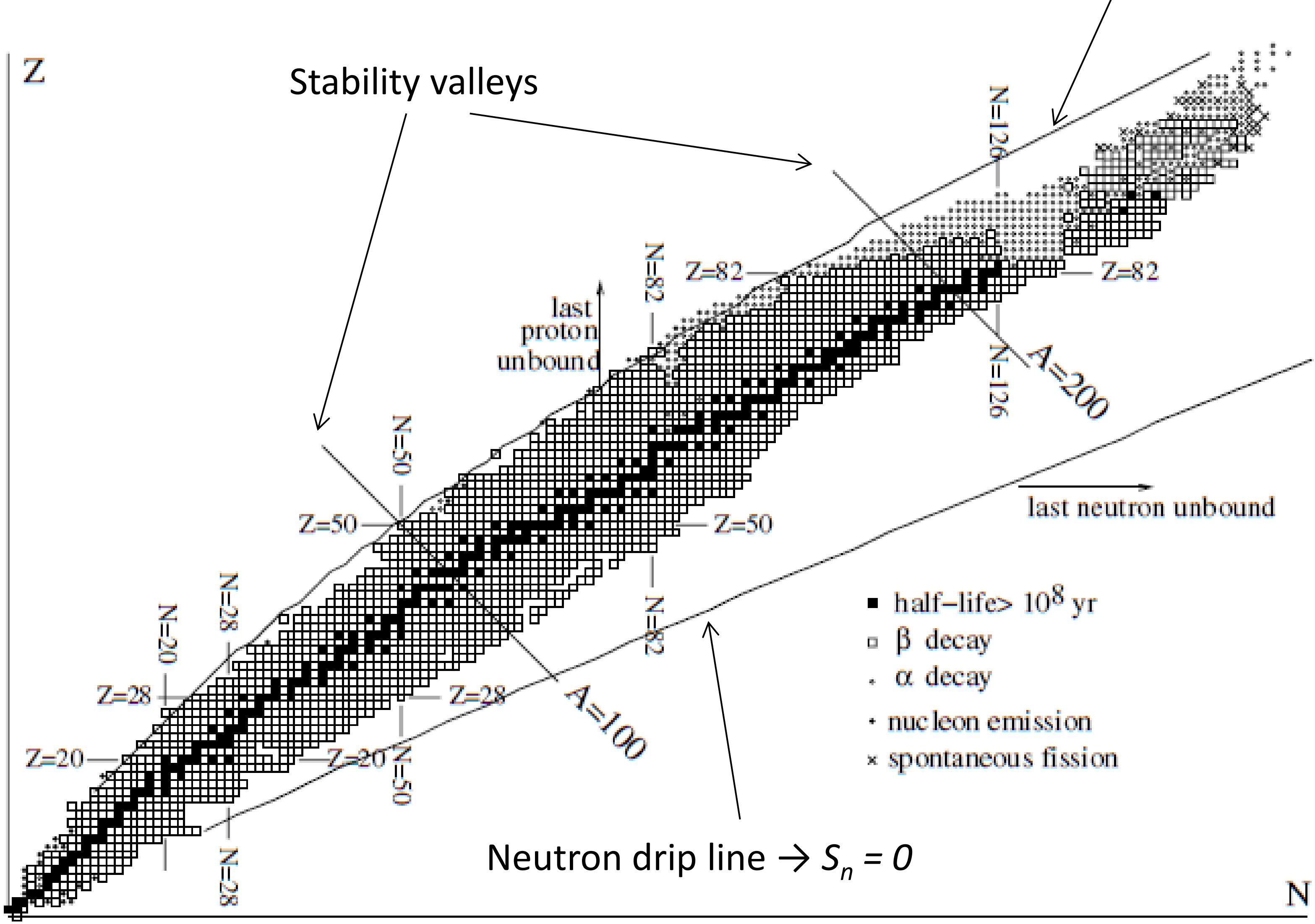


- For N and $Z < 20$ → stable nuclei close to the straight line $N = Z$ (only ^{3}He is upper)
- Tc (Z = 43) and Pm (Z = 61) have no stable isotope
- For N and $Z > 20$ → stable isotopes move away from $N = Z$ line → increasing effect of Coulomb repulsion → for last stable nuclei $N/Z = 1.5$

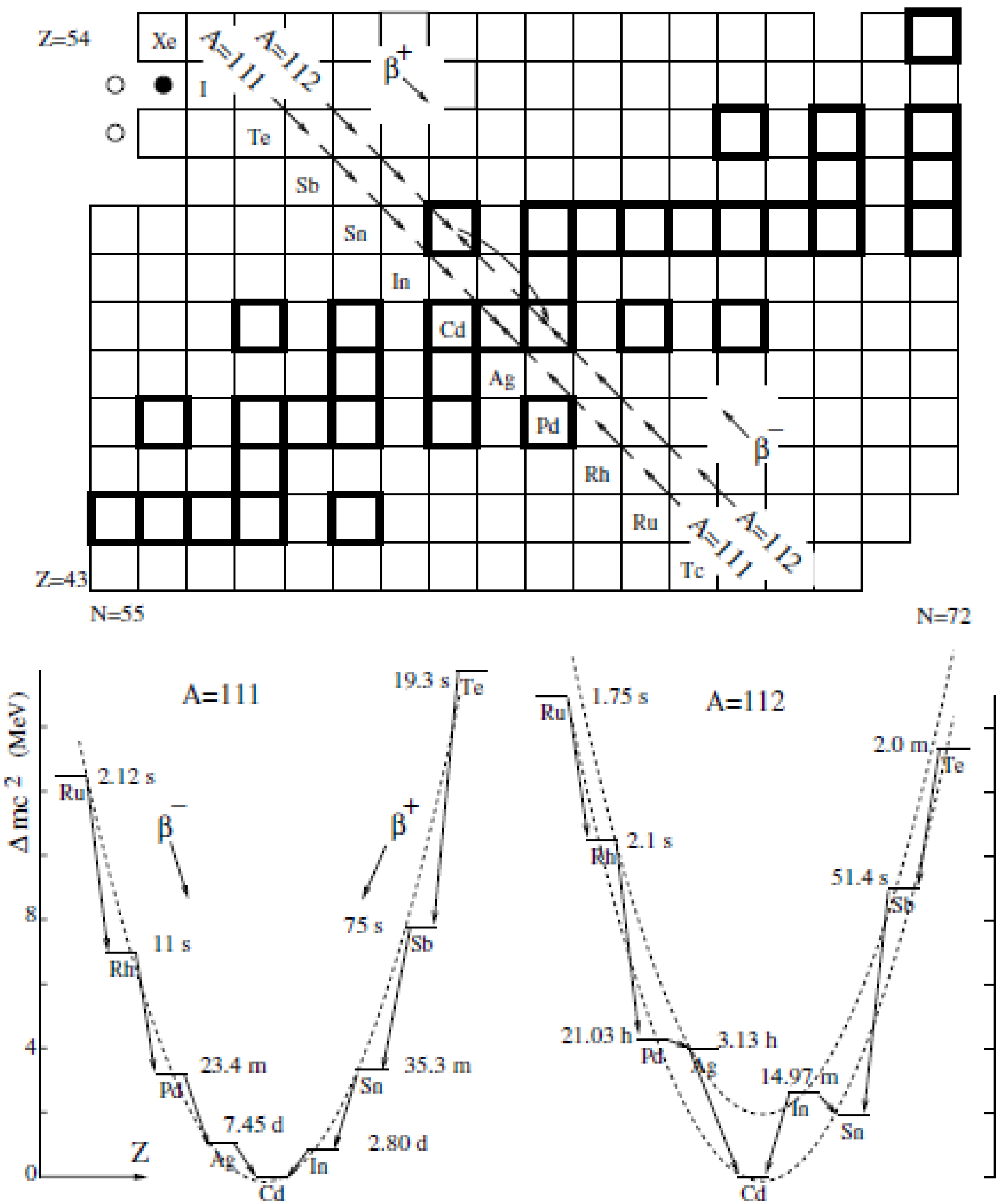
Stable nuclei (2)



Stable nuclei (3)



Stable nuclei (4)



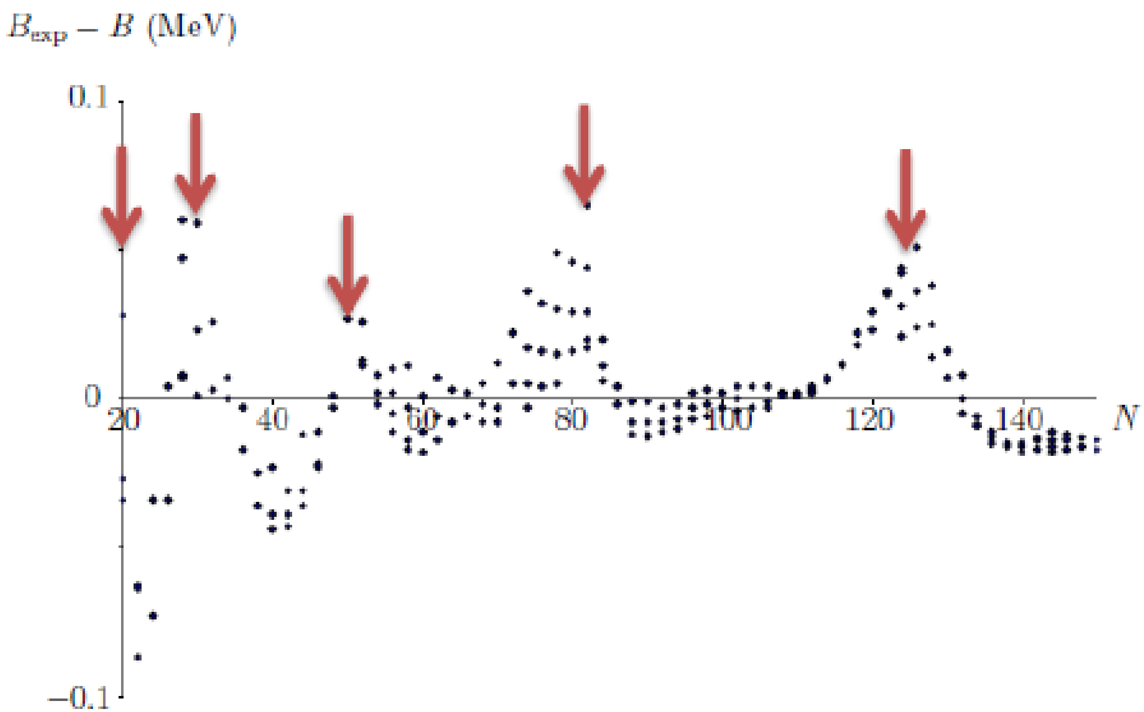
- Nuclei with an excess of neutrons (below the β stable nucleus) decay via β^- emission
- Nuclei with an excess of protons (above the β stable nucleus) decay via β^+ emission or electron capture
- The dashed lines show the predictions of the Bethe-Weizsäcker formula
- Note that for even-*A* → two stable isobars ^{112}Sn and ^{112}Cd

Magic numbers (1)

- In atomic physics → the ionization energy I (the energy needed to extract an electron from a neutral atom) shows discontinuities around Z = 2, 10, 18, 36, 54 and 86 (i.e. for noble gases) → these discontinuities are associated with closed electron shells
- An analogous phenomenon occurs in nuclear physics → there exist many experimental indications showing that atomic nuclei possess a shell-structure → they can be constructed (like atoms) by filling successive shells of an effective potential well (shell model)
- Separation energies present discontinuities at special values of N or Z → these numbers are called *magic numbers*
- These numbers are 2 – 8 – 20 – 28 – 50 – 82 – 126
- The discontinuity in the separation energies is due to the excess binding energy for magic nuclei as compared to that predicted by the Bethe-Weizsäcker formula

Magic numbers (2)

- For the same reason (shell effect not considered in the liquid drop model) → the difference between the experimental binding energy B_{exp} and the binding energy B calculated from the Bethe-Weizsäcker formula is maximum for these N values

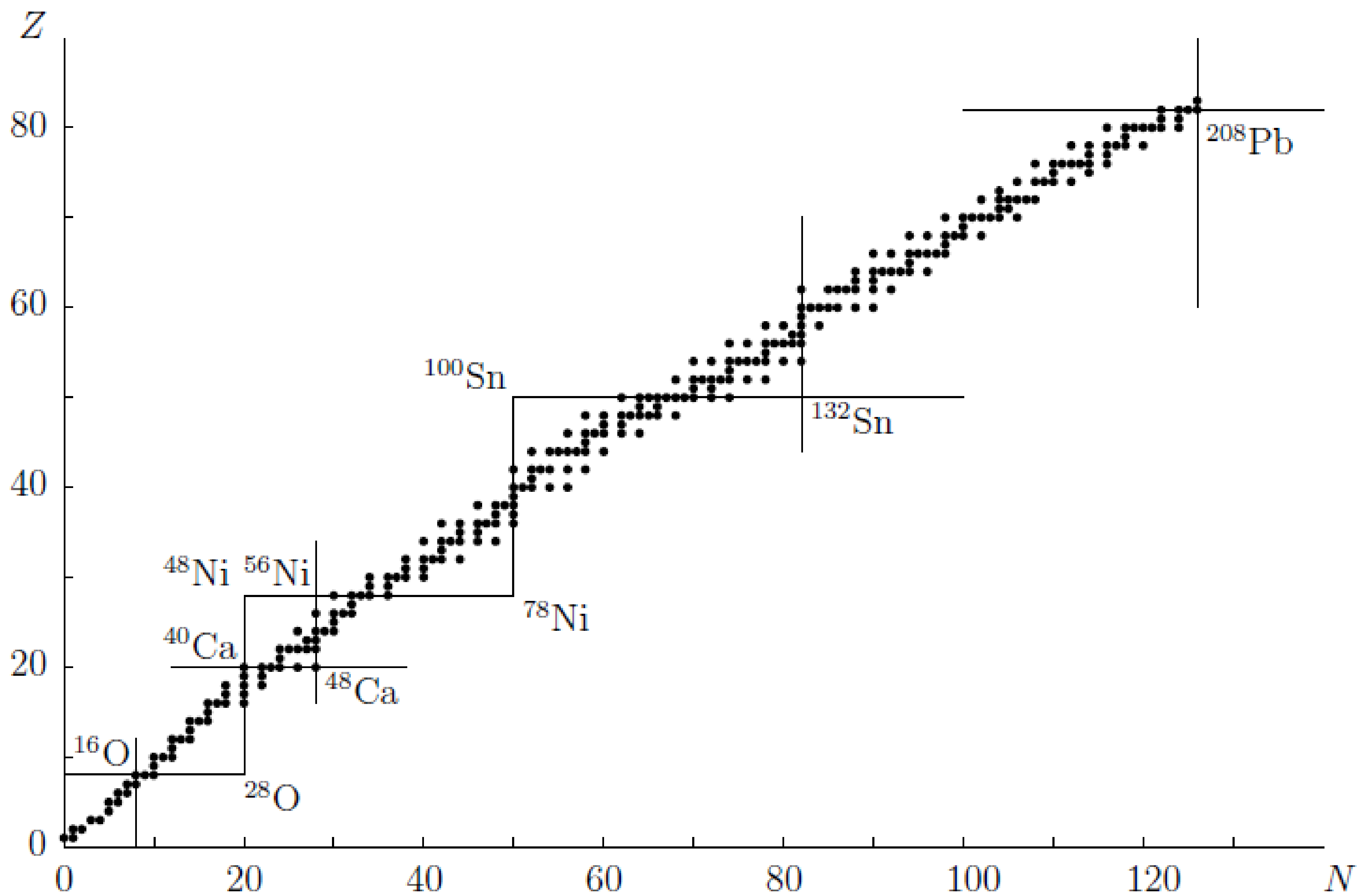


This difference is also observed as a function of Z

Magic numbers (3)

- Equivalent variations or discontinuities appear for other quantities as radius, electric and magnetic moments,…
- Nuclei with magic numbers of neutrons or protons have a closed shell that encourages a spherical shape
- Nuclei having both magic neutrons and protons are particularly stable → they are called *doubly magic* nuclei → ^{4}He (Z = 2), ^{16}O (Z = 8), ^{40}Ca (Z = 20), ^{208}Pb (Z = 82)
- Some nuclei having magic number can be instable → but generally with a radioactivity smaller than waited → ^{28}O (Z = 8), ^{48}Ni, ^{56}Ni, ^{78}Ni (Z = 28), ^{100}Sn, ^{132}Sn (Z = 50)
- The following magic number (not observed) could be 184

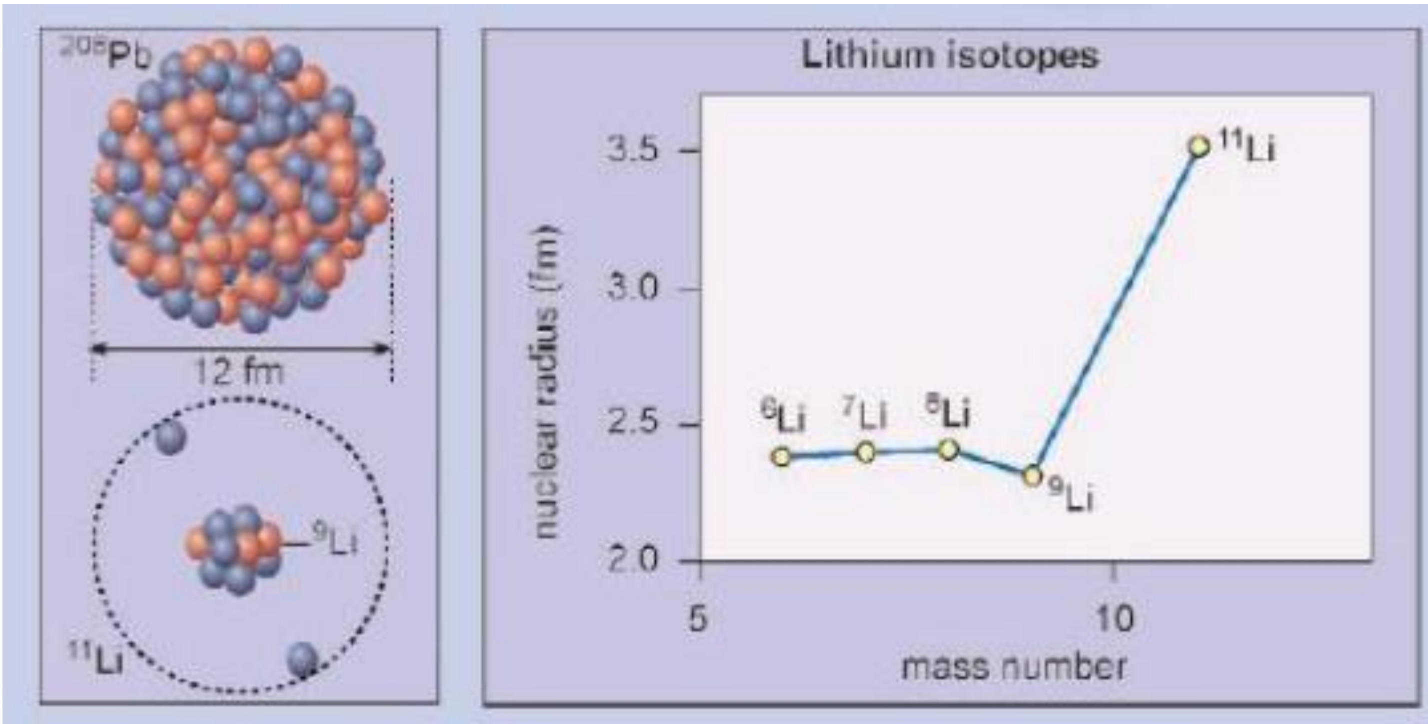
Magic numbers (4)



Particular types of nuclei (1)

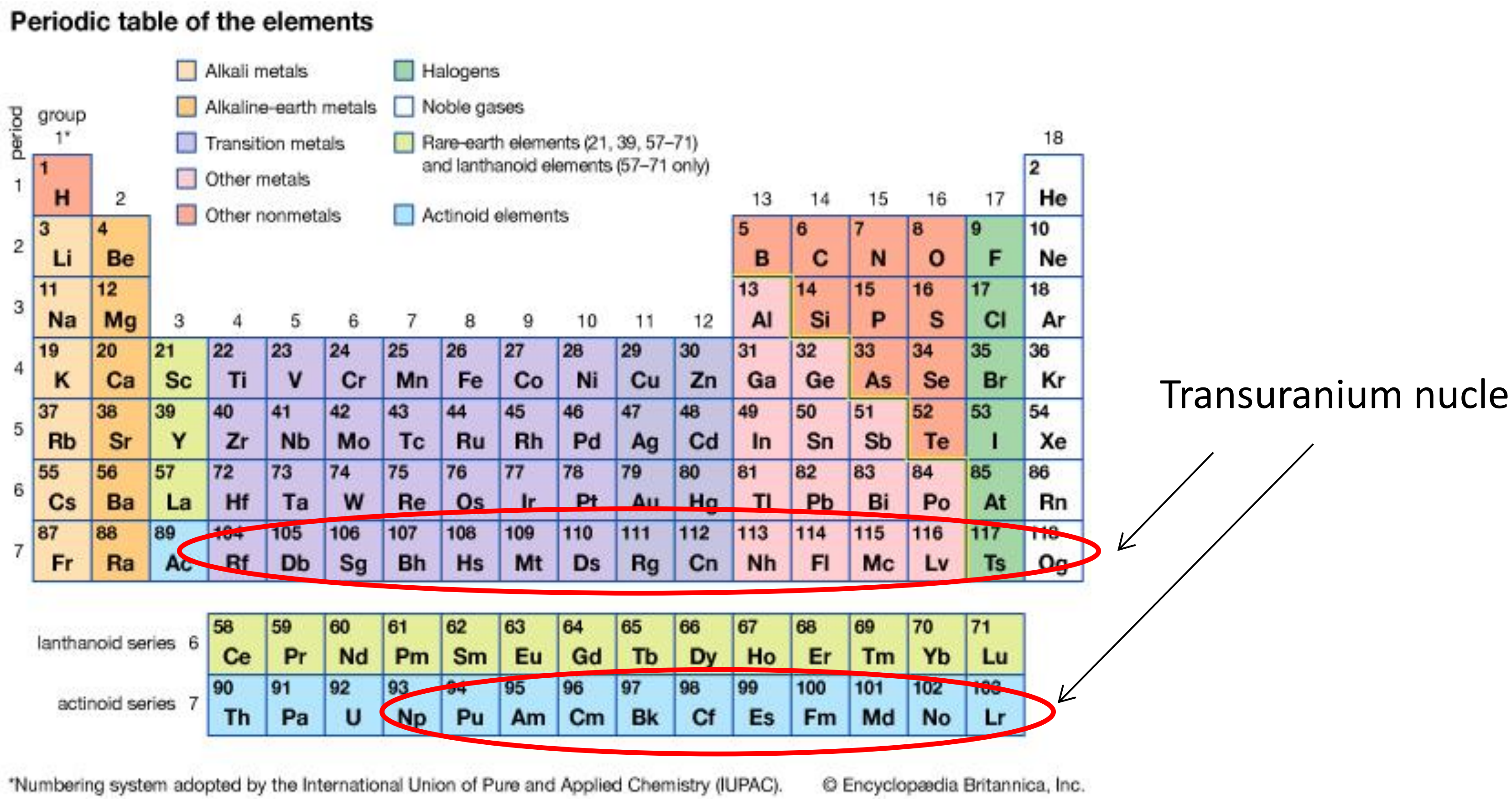
- Exotic nuclei: Instable nuclei characterized by a number of neutrons or protons vary far from the stability valley (examples: ^{24}O,…)
- Halo nuclei: Exotic nuclei with a radius appreciably larger than that predicted by the rule $R = R_0 A^{-1/3}$ → they are characterized by a core nucleus (with normal radius) surrounded by a halo of orbiting protons or neutrons → they have necessarily very weak separation energy (examples: ^{8}He, ^{22}C …)
- Transuranium nuclei (also called transuranic nuclei) nuclei with atomic number greater than 92 (*Z* of uranium, last natural element):

Particular types of nuclei (2)



Example of halo nuclei: ^{11}Li

Particular types of nuclei (3)



- Superheavy nuclei: Hypothetic nuclei with life time larger than the last transuranium nuclei due to the proximity of the next magic number (126) → island of stability