Chapter VI:
Alpha decay

Summary

1. General principles

2. Energy and momentum conservations

3. Theory of α emission

4. Angular momentum and parity

General principles (1)

- The nucleus emit an α particle i.e. a nucleus of helium: ${}^4_2\mathrm{He}_2$
- Alpha emission is a Coulomb repulsion effect → becomes important for heavy nuclei (Coulomb repulsion in heavy nuclei due to the larger number of protons present) → Bethe-Weizsäcker formula → Coulomb force ↗ with size at a faster rate (in $\sim Z^2$) than does the specific nuclear binding force (in $\sim A$)
- The α emission is particularly favored (compared to other particles) due to
  - Its very stable and tightly bound structure
  - Its small mass
  - Its small charge
- Theoretically some heavier particles as ^{8}Be or ^{12}C may be emitted or the fission into equal daughter nuclides may happen → but very penalized

General principles (2)

- For a nucleus to be an α emitter → not enough for the α decay to be energetically possible → the disintegration constant must also not be too small → the α emission would occur so rarely that it may never be detected ($T_{½}$ < 10^{16} years)
- Moreover if β decay is present → can mask the α decay
- Most nuclei with A > 190 and many with 150 < A < 190 are energetically possible emitters but ½ of them effectively meet the 2 other conditions

Energy conservation (1)

- The α emission process (between ground state levels) is:

$$^{A}_{Z}X_{N} \rightarrow {}^{A-4}_{Z-2}X'_{N-2} + \alpha$$

- Rutherford shows in 1908 that the α particle is a nucleus of ^{4}He → constituted of 2 protons and 2 neutrons
- Energy conservation with the initial decaying nucleus X at rest →

$$m_X c^2 = m_{X'} c^2 + T_{X'} + m_\alpha c^2 + T_\alpha$$

- Due to the linear momentum conservation → X' and α are in motion → *T* is the kinetic energy
- Equivalently we write →

$$T_{X'} + T_\alpha = (m_X - m_{X'} - m_\alpha)c^2$$

Energy conservation (2)

- $Q = (m_X - m_{X'} - m_\alpha)c^2$ = the net energy released in the decay → the decay occurs spontaneously only if $Q > 0$
- Q can be calculated from atomic masses (even we discuss about nuclear processes) because the electron masses cancel in the subtraction
- For a typical α emitter (232-U) → Q may be calculated from the known masses for various emitted particles:

Emitted Particle	Energy Release (MeV)	Emitted Particle	Energy Release (MeV)
n	−7.26	^{4}He	+5.41
^{1}H	−6.12	^{5}He	−2.59
^{2}H	−10.70	^{6}He	−6.19
^{3}H	−10.24	^{6}Li	−3.79
^{3}He	−9.92	^{7}Li	−1.94

Energy conservation (3)

- Only α emission is possible in this case → α is very stable → α has a relatively small mass compared with the mass of its separate constituents
- The Q value is also the total kinetic energy given to the decay fragments $Q = T_{X'} + T_{\alpha}$
- For $Q > 0$ → we find back the condition $m_X > m_{X'} + m_{\alpha}$ of instability in particles
- Remark: α disintegration towards excited levels of X' are possible → the excitation energy of the nucleus X' has to be subtracted from Q

Linear momentum conservation

- As the original nucleus is at rest → X' and α move with equal and opposite momenta → $p_\alpha = p_{X'}$
- As the α decay released typically 5 MeV → we can use nonrelativistic kinematics → $m_\alpha T_\alpha = m_{X'} T_{X'}$ →

$$T_\alpha = \frac{Q}{1 + m_\alpha / m_{X'}}$$

- As X' is a heavy nucleus → A $\gg$ 4 →

$$T_\alpha = Q(1 - 4/A), \;\; T_{X'} = 4Q/A$$

- Typically the α carries 98% of the *Q* energy and X' carries 2% (recoil energy) corresponding for α = 5 MeV to $T_{X'}$ = 100 keV

Released energy (1)

- Introducing the binding energies the energy released during the α decay may be written →

$$Q = B(4,2) + B(A-4, Z-2) - B(A,Z)$$

- Thus $Q > 0$ becomes →

$$\begin{aligned} B(4,2) \quad &> \quad B(A,Z) - B(A-4,Z-2) \\ &= \quad A\frac{B(A,Z)}{A} - (A-4)\frac{B(A-4,Z-2)}{A-4} \\ &= \quad 4\left(\frac{B(A,Z)}{A}\right)_m + (A-2)\Delta \end{aligned}$$

- $(B/A)_m$ is the mean binding energy by nucleon of parent and daughter nuclei and Δ is their difference (≈ 30 keV for heavy nuclei)

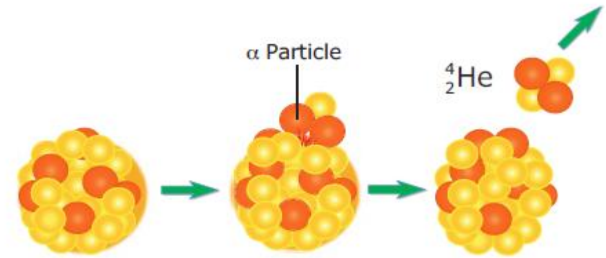
Released energy (2)

- As $B(4,2) \approx 28$ MeV → we have thus approximately →

$$\left(\frac{B(A,Z)}{A}\right)_m < \frac{B(4,2)}{4} \approx 7\ \mathrm{MeV}$$

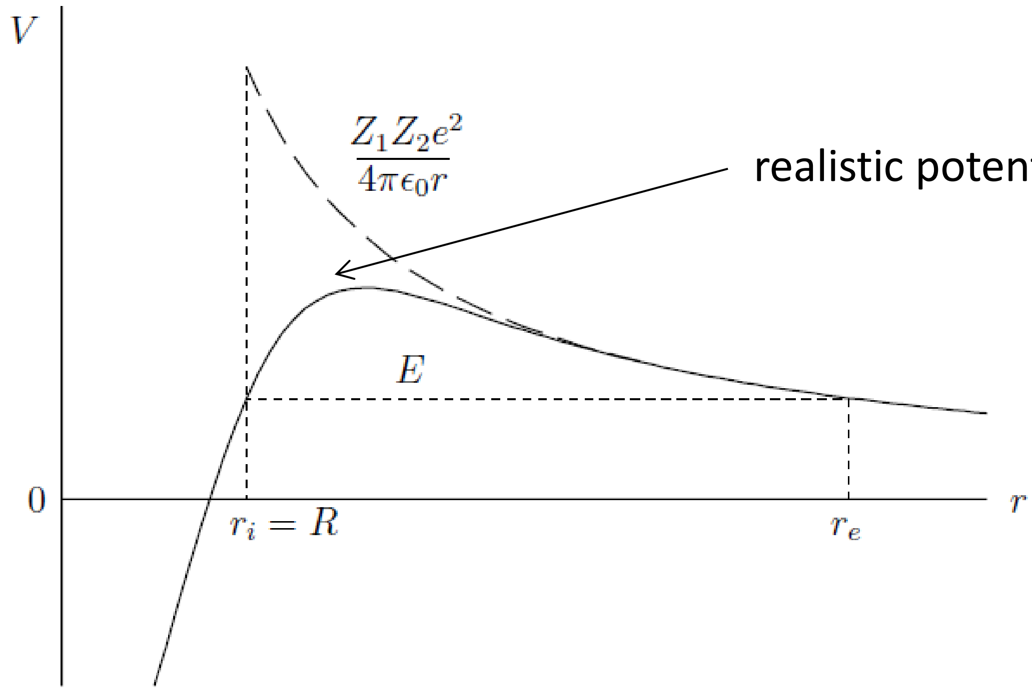
- Du to the term Δ → α decay becomes frequent for nuclei with *A* > 200 for which *B/A* is < 7.8 MeV
- This also explains why the α emission is favored compared to other nuclei as deuteron (B(2,1)/2 ≈ 1.11 MeV) or tritium (B(3,1)/3 ≈ 2.83 MeV) → indeed the α particle has tightly bound structure → a pair of neutrons and a pair of protons inside a nucleus is favored to form an α-cluster

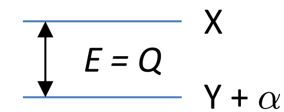
Theory of α emission (1)



- General features of α emission theory have been developed by Gamow, Gurney and Condon in 1928
- The α particle is assumed to move in a spherical region determined by the daughter nucleus → one-body model
- The α particle is preformed inside the parent nucleus → there is no proof that it is well the case but it works quite well
- Deeply inside the heavy nucleus → attractive nuclear force dominates the Coulomb repulsion force
- Outside the nucleus → only remains the Coulomb force
- Between the 2 (close to the nucleus surface) → number of neighbours ↘ → nuclear attractive force ↘ → equilibrium with repulsion force → potential barrier
- To be emitted the α particle has to cross this potential barrier by tunnel effect

Theory of α emission (2)

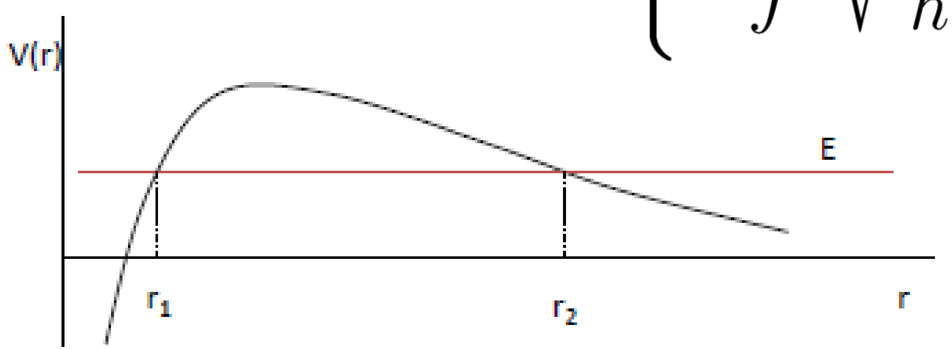




- The probability of disintegration per time unit *W* is supposed to be $\propto$ to the probability of crossing the barrier → $W \propto |T|^2$ with T the transmission coefficient
- The transmission probability is given by the WKB approximation

WKB approximation

- For decay between ground state levels (e.g. 0^+) → we have to resolve $u''(r) - 2\mu/\hbar^2\,(V(r) - E)\,u(r) = 0$ (with $\mu \approx m_\alpha m_{X'}/m_X$ the reduced mass of the α nucleus and the daughter nucleus)
- By choosing $u(r) = \exp(iS(r))$ with $S(r)$ slowly varying → $S(r)'^2 = 2\mu/\hbar^2\,(E - V(r))$
- Si $E < V(r)$ → $$u(r) = \exp\left\{-\int\sqrt{\frac{2\mu}{\hbar^2}(V(r)-E)dr}\right\}$$



G = Gamow factor

$$|T|^2 = \left|\frac{u(r_2)}{u(r_1)}\right|^2 = \exp\left\{-2\int_{r_1}^{r_2}\sqrt{\frac{2\mu}{\hbar^2}(V(r)-E)dr}\right\} = \exp\left(-2G\right)$$

Theory of α emission (3)

- WKB approximation applied to α emission → $r_1 = R$ (the radius of the nucleus), $r_2 = 2(Z\text{-}2)e^2/4\pi\epsilon_0 E$, with

$$V(r) = \begin{cases} 0 & r < R \\ 2(Z-2)e^2/4\pi\epsilon_0 r & r \geq R \end{cases}$$

$$\Rightarrow$$

$$|T|^2 = \exp\left\{-2\left(\frac{2\mu E}{\hbar^2}\right)^{1/2}\int_R^{r_2}\left(\frac{r_2}{r}-1\right)^{1/2}dr\right\}$$

- Transforming $r = r_2\cos^2 u$ →

$$\int_R^{r_2}\left(\frac{r_2}{r}-1\right)^{1/2}dr = r_2\left[\arccos\sqrt{\frac{R}{r_2}} - \sqrt{\frac{R}{r_2}}\sqrt{1-\frac{R}{r_2}}\right]$$

Theory of α emission (4)

- For a heavy nucleus emitting 5 MeV α → $r_2 \approx 0.6\,Z$ fm > $R \approx 1.25\,A^{1/3}$ fm
- Assuming $R/r_2 \approx 0$ → we obtain the Gamow approximation →

$$|T|^2 \approx \exp(-2\pi\eta)$$

where η the Sommerfeld factor is (α = 1/137)

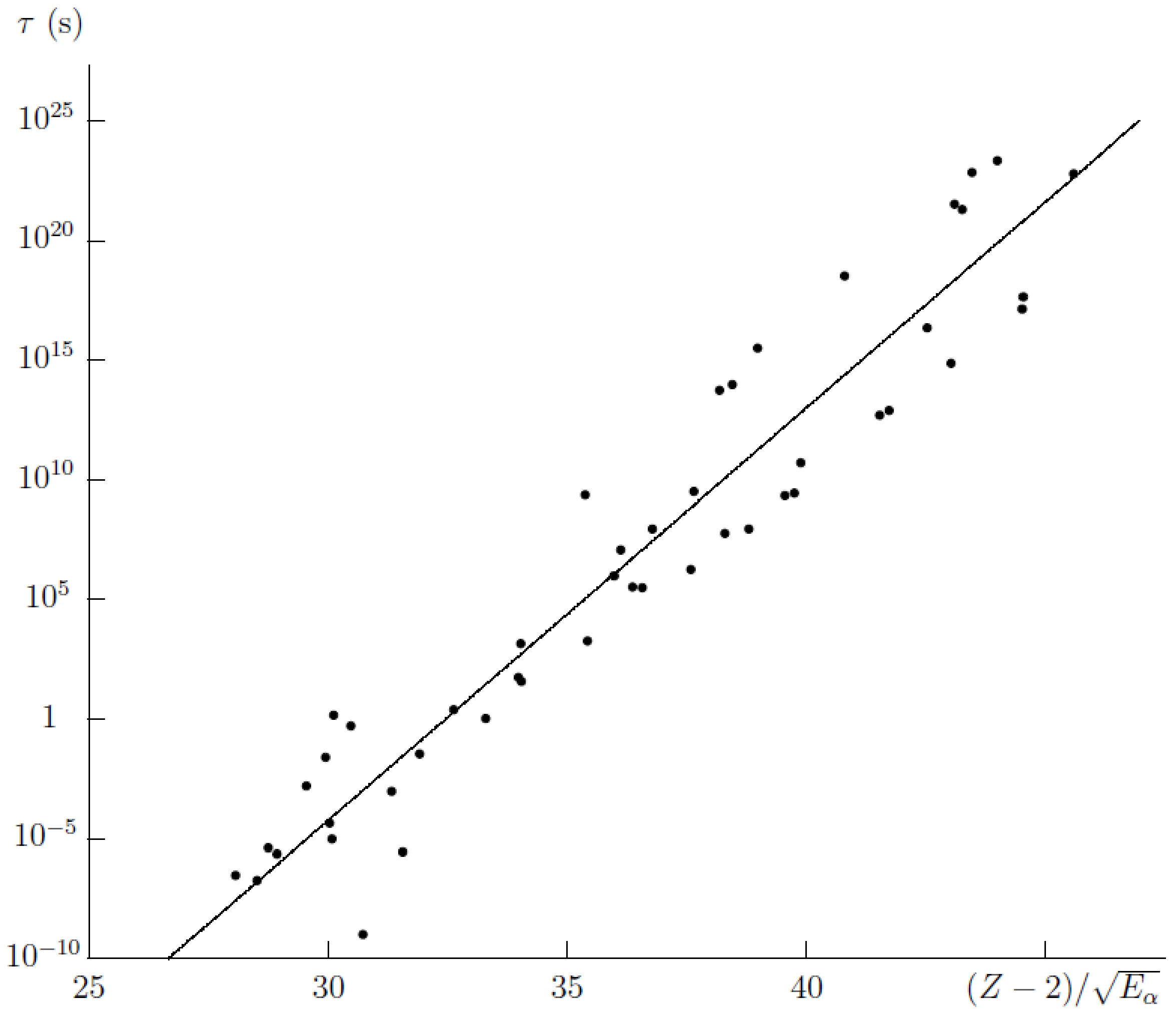
$$\eta = (Z-2)\alpha\sqrt{\frac{2\mu c^2}{E}}$$

- The mean lifetime $\tau = W^{-1}$ approximately follows →

$$\log_{10}\tau \approx C_1 + C_2\frac{Z-2}{\sqrt{E}}$$

$$\text{with } C_2 = 2\pi(\log_{10} e)\alpha\sqrt{2\mu c^2}$$

Theory of α emission (5)



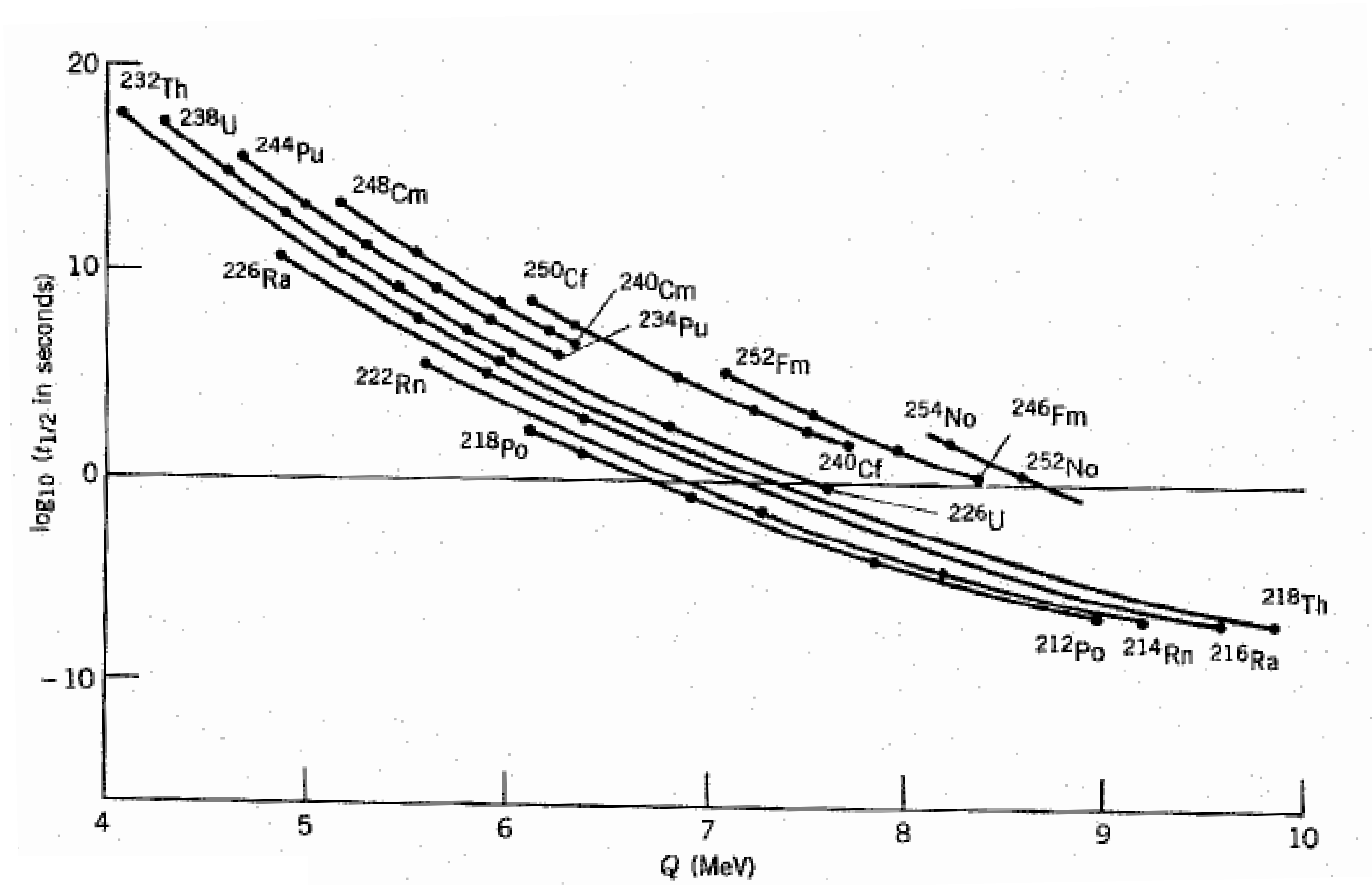
E_α is the kinetic energy of the α particle in MeV

Theory of α emission (6)

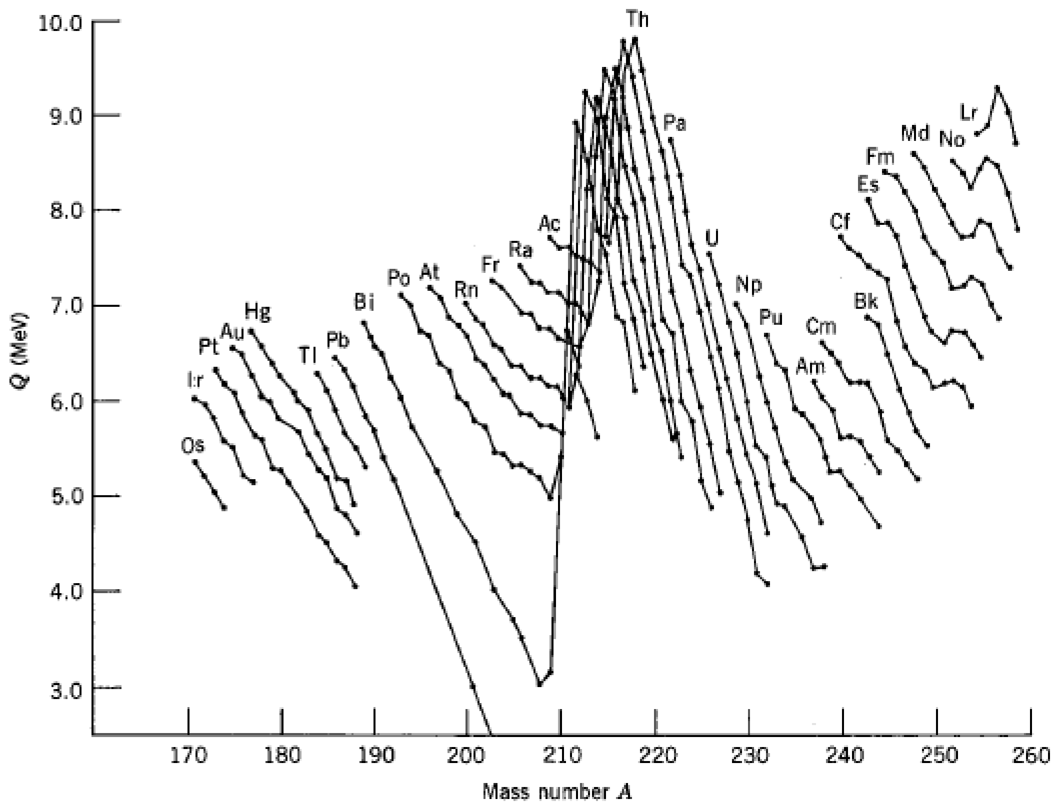
- The α emitters with short mean lifetime have large disintegration energy (and conversely) $\leftrightarrow$ observed in 1911: Geiger-Nutall rule
- Examples: 232-Th: $T_{½}$ = 1.4 $\times$ 10^{10} years, Q = 4.08 MeV and 218-Th: $T_{½}$ = 1.0 $\times$ 10^{-7} s, Q = 9.85 MeV
- A factor 2 in energy implies a factor 10^{24} in half-life
- Correct tendency for all isotopes but very good only for Z and N both even nuclei

		$t_{1/2}$ (s)	
A	Q (MeV)	Measured	Calculated
220	8.95	10^{-5}	3.3×10^{-7}
222	8.13	2.8×10^{-3}	6.3×10^{-5}
224	7.31	1.04	3.3×10^{-2}
226	6.45	1854	6.0×10^{1}
228	5.52	6.0×10^{7}	2.4×10^{6}
230	4.77	2.5×10^{12}	1.0×10^{11}
232	4.08	4.4×10^{17}	2.6×10^{16}

Theory of α emission (7)



Theory of α emission (8)



- When A > 212 → adding neutrons to a nucleus reduces the disintegration energy → due to the Geiger-Nuttall rule → increase of the half-life → the nucleus becomes more stable
- The discontinuity near A = 212 occurs where N = 126 → another example of nuclear shell structure.

Theory of α emission (9)

- Geiger-Nutall rule enables us to understand why other decays into light particles are not commonly seen (even though they are allowed by the Q value)
- Geiger-Nutall rule is able to reproduce $T_{½}$ within 1-2 orders of magnitude over a range of more than 20 orders
- Approximations in previous calculations:

1. Initial and final wave functions ($\leftrightarrow$ Fermi Golden Rule) are not considered
2. The nucleus is assumed to be spherical with $R \approx 1.25\, A^{1/3}$ fm $\rightarrow$ heavy nuclei (specially with $A \geq 230$) have strongly deformed shape
3. The angular momentum carried by the α particle is neglected

Angular momentum and parity (1)

- In previous calculations → transition between ground state levels (e.g. 0^+) → but an initial state can populate different final states in the daughter nucleus → « fine structure » of α decay

- In that case we have to consider the angular momentum J_i and J_f of the of the initial and final nuclear state → and consequently the angular momentum of the α particle $\boldsymbol{\ell}_\alpha$

- Consequently → α decay must follow the laws of the conservation of angular momentum and of parity

Angular momentum and parity (2)

- Definition of the total angular momentum **J** for *i* nucleons:

$$\boldsymbol{J} = \sum_{i=1}^{A} (\boldsymbol{L}_i + \boldsymbol{S}_i)$$

with $\boldsymbol{L}_i$ and the orbital angular momentum operator and $\boldsymbol{S}_i$ the spin operator of the i^{th} nucleon

- In the particular case of α decay → this expression may be written (with $\boldsymbol{I}_\alpha$ and $\boldsymbol{\ell}_\alpha$ the spin and angular momentum of the α particle and $\boldsymbol{J}_i$ and $\boldsymbol{J}_f$ written for initial and final nuclear states) →

$$\boldsymbol{J}_i = \boldsymbol{J}_f + \boldsymbol{I}_\alpha + \boldsymbol{\ell}_\alpha$$

- The α particle wave function is then represented by a $Y_{\ell m}$ with $\ell = \ell_\alpha$

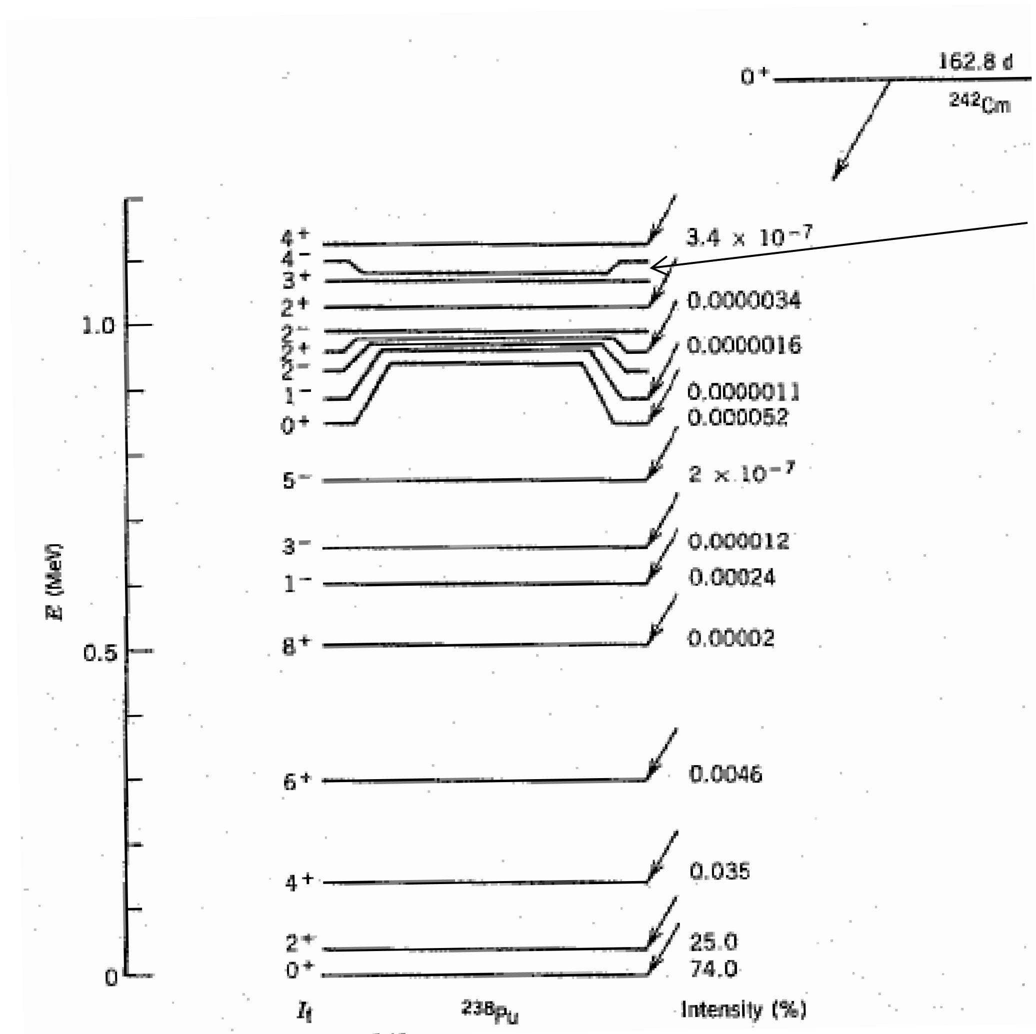
Angular momentum and parity (3)

- As the ^{4}He nucleus consists of 2 protons and 2 neutrons all in 1s state → their spins coupled pairwise → $I_\alpha = 0$
- The composition of the 3 remaining angular momenta leads to →
$$|J_i - J_f| \leq \ell_\alpha \leq J_i + J_f$$
- The conservation of parity implies →
$$\pi_i = \pi_f \pi_\alpha (-1)^{\ell_\alpha}$$
- Moreover as the parity π_α of α particle is + (even-even nucleus) → the parity conservation rule becomes →
$$(-1)^{\ell_\alpha} = \pi_i \pi_f$$
- If the initial and final parities are the same → ℓ_α must be even ↔ If the parities are different → ℓ_α must be odd
- In particular for an initial state 0^+ (frequent case) → $\ell_\alpha = J_f$

Angular momentum and parity (4)

- Another consequence of the introduction of ℓ_α → the barrier of potential is raised and becomes (particle in a well):
$$V(r) + \hbar^2 \ell_\alpha(\ell_\alpha + 1)/2mr^2$$
- The additional term is always > 0 → ↗ of the barrier thickness → the probability transition ↘
- Moreover the *Q* value ↘ when the final state is not the ground state: $Q \rightarrow Q - E_x$ with E_x the energy of the excited state → application of the Geiger-Nutall rule → a smaller *Q* value implies a large mean lifetime → a small transition probability → a small intensity in the decay branch
- These 2 reasons implies a ↘ of the probability transition when the final state is not the ground state

Angular momentum and parity (5)



The 3^+ state is forbidden by the parity selection rule → 0 $\rightarrow$ 3 decay must have ℓ_α = 3 → the parity has to change

Angular momentum and parity (6)