A new level-dependent coarse grid correction scheme for indefinite Helmholtz problems

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Fast and efficient numerical solvers for indefinite Helmholtz problems are of great interest in many scientific domains that study acoustic, seismic or electromagnetic wave scattering. Applications such as engine design, oil exploration, medical imaging, but even quantum mechanical problems describing particle interaction [1], are governed by underlying Helmholtz equations of the form

$$Hu(\mathbf{x}) = (-\Delta - k(\mathbf{x})^2)u(\mathbf{x}) = f(\mathbf{x}), \qquad \text{with } \mathbf{x} \in \mathbb{R}. \tag{1}$$

Pushed by the rising interest in high resolution requirements and high-dimensional applications, the diffusion term in the Laplacian equation drives the condition number of the associated discretized operator to undesirable sizes for standard iterative methods to converge rapidly. In addition, for realistic values of the wavenumber $k(\mathbf{x})$ in (1), the Helmholtz operator H becomes indefinite, destroying the convergence behaviour of much preferred sparse linear system solvers such as e.g. Krylov subspace methods and classical geometric multigrid.

When the negative shifting term $-k(\mathbf{x})^2$ in the Helmholtz operator in (1) is replaced by a complex valued shift $-(\beta_1+\imath\beta_2)k(\mathbf{x})^2$ the resulting operator is still closely related to the original, yet can efficiently be inverted with e.g. standard multigrid methods. This idea defined a well-known and successful Helmholtz preconditioning technique called complex shifted Laplacian [2]. The choice of the optimal value of the scaling parameter $\beta_1 + \imath\beta_2$ is a trade-off between a good preconditioner on the one hand and a computationally cheap inversion of that preconditioner on the other hand.

Inspired by the complex shifted Laplacian, we present the construction and analysis of a modified multigrid method that is capable of solving the original indefinite Helmholtz equation (1) on the finest grid using a series of multigrid cycles with a level-dependent complex shift, i.e. gradually perturbing the original Helmholtz operator throughout the hierarchy, leading to a stable correction scheme on all levels.

It is rigorously shown that the adaptation of the complex shift throughout the multigrid cycle maintains the functionality of the two-grid correction scheme, as no smooth modes are amplified in or added to the error. Complementary, a sufficiently smoothing relaxation scheme should be applied to ensure damping of the oscillatory error components. Contrary to classical multigrid preconditioning techniques like shifted Laplacian, the proposed level-dependent multigrid scheme is capable of directly solving the Helmholtz system (1) instead of being used as a preconditioner.

Numerical experiments on various physically relevant benchmark problems show the level-dependent multigrid solver to be competitive with or even outperform contemporary multigrid-preconditioned Krylov methods that use the classical level-fixed complex shift preconditioner.

References

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