Optimal-order multigrid preconditioners for linear systems arising in the semi-smooth Newton solution of certain PDE-constrained optimization problems

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We present a new technique for constructing multigrid preconditioners arising in the semi-smooth Newton solution process of optimization problems of the form

$$\min_{u\in L^2(\Omega)} \frac{1}{2}\|\mathcal{K}u-b\|^2+\frac{\beta}{2}\|u\|^2\ ,\quad \underline{u}\le u\le \overline{u}\ , \tag{1}$$

where $\mathcal{K}: L^2(\Omega)\to\mathcal{Y}$ is a bounded linear operator, with the embedding $\mathcal{Y}\hookrightarrow L^2(\Omega)$ being compact, and $\Omega\subset\mathbb{R}^d$, $d=1,2,3$, is bounded domain. Problem (1) can be regarded as the reduced form of a PDE-constrained optimization problem with $\mathcal{K}$ being the solution operator of a PDE (for example, $\mathcal{K}=(-\Delta)^{-1}: L^2(\Omega)\to H^1_0(\Omega)$).

For a piecewise constant discretization of the discrete control space V_h, each semi-smooth Newton (outer) iteration requires the solution of a linear system whose matrix is a principal submatrix of $G_h=K_h^TK_h+\beta I$, where K_h is the matrix representing the discretization of $\mathcal{K}$, and h is the mesh size. In a large-scale context these (inner) linear systems are solved using preconditioned conjugate gradient. An earlier technique [1] produced a multigrid preconditioner M_h for G_h that satisfies, under reasonable conditions

$$1-C\frac{h^{\frac{1}{2}}}{\beta}\le\frac{\langle M_hu,u\rangle}{\langle G_hu,u\rangle}\le 1+C\frac{h^{\frac{1}{2}}}{\beta},\quad \forall u\in V_h\setminus\{0\}\ . \tag{2}$$

As a result of (2), the number of inner linear iterations needed to solve the system at each outer iteration decreases with $h\downarrow 0$. While this result is interesting from a theoretical point of view (and qualitatively consistent with the behavior of the preconditioner for the full system), its practicality is limited by the suboptimal factor $h^{1/2}$ in (2).

The new technique, relying on constructing larger and non-conforming coarse spaces, produces multigrid preconditioners M_h that are able to capture the character of the operator G_h in an optimal way, namely we have

$$1-C\frac{h}{\beta}\le\frac{\langle M_hu,u\rangle}{\langle G_hu,u\rangle}\le 1+C\frac{h}{\beta},\quad \forall u\in V_h\setminus\{0\}\ .$$

References

[1] Andrei Drăgănescu, *Multigrid Preconditioning of Linear Systems for Semismooth Newton Methods Applied to Optimization Problems Constrained by Smoothing Operators*, Optimization Methods and Software (DOI:10.1080/10556788.2013.854356), 2013.